General Relativistic Static Fluid Solutions with Cosmological Constant

Diplomarbeit

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Contents

1 Introduction

This diploma thesis analyses static, spherically symmetric perfect fluid solutions to Einstein's field equations with cosmological constant. New kinds of global solutions are described.

By a global solution one means an inextendible spacetime satisfying the Einstein equations with cosmological constant with a perfect fluid source. The matter either occupies the whole space or has finite extend. In the second case a vacuum solution is joined on as an exterior field.

Global static fluid ball solutions with finite radius at which the pressure vanishes are called stellar models.

Recent cosmological observations give strong indications for the presence of a positive cosmological constant with $\Lambda < 3 \times 10^{-52}\mathrm{m}^{-2}$. On the other hand Anti-de Sitter spacetimes, having negative cosmological constant, are important in the low energy limit of superstring theory.

Therefore it is interesting to analyse solutions to the field equations with cosmological constant representing for example relativistic stars.

Vanishing cosmological constant $\Lambda = 0$

The first static, spherically symmetric perfect fluid solution with constant density was already found by Schwarzschild in 1918.

In spherical symmetry Tolman [25] and Oppenheimer and Volkoff [18] reduced the field equations to the well known TOV equation.

The boundary of stellar models is defined to be where the pressure vanishes. At this surface a vacuum solution is joined on as an exterior field. In case of vanishing cosmological constant it is the Schwarzschild solution. For very simple equations of state Tolman integrated the TOV equation and discussed solutions. Although he already included the cosmological constant in his calculations he did not analyse them. He stated that the cosmological constant is too small to produce effects.

Buchdahl [2] assumed the existence of a global static solution, to show that the total mass of a fluid ball is bounded by its radius. He showed the strict inequality $M < (4/9)R$, which holds for fluid balls in which the density does not increase outwards. It implies that radii of fluid balls are always larger than the black-hole event horizon.

Geometrical properties of constant density solutions were analysed by Stephani [21, 22]. He showed that they can be embedded in a five dimensional flat space and that they are conformally flat. The cosmological constant can easily be included in his calculations by redefining some variables.

Conformal flatness of constant density solutions is easily shown with the use of new Buchdahl variables, which will be introduced in the fourth chapter. It will be seen that the field equations imply the vanishing of the squared Weyl tensor.

Moreover he remarked that constant density solutions without cosmological constant have the spatial geometry of a 3-sphere. With cosmological constant the spatial part of the metric may also be Euclidean or hyperbolic, depending on the choices of constant density and cosmological constant.

As already said, Buchdahl assumed the existence of a global static solution to derive the upper bound on the mass for given radius. Rendall and Schmidt [19] proved the existence of such a global static solution. Baumgarte and Rendall [1] later improved the argument with less assumptions on the equation of state and the pressure. This was further improved by Mars, Martín-Prats and Senovilla [14].

There is a conjecture that asymptotically flat static perfect fluid solutions are spherically symmetric. If this is true then this means that spherically symmetric solutions are the most general static perfect fluid models.

Non-vanishing cosmological constant $\Lambda \neq 0$

This diploma thesis proves existence of a global solution for cosmological constants satisfying $\Lambda < 4\pi\rho_b$. ρ_b denotes the boundary density and is given by the equation of state. For a degenerate neutron star one may assume the boundary density to be the density of iron. In physical units this leads to $\Lambda < (4\pi G/c^2)\rho_{\mathrm{Fe}} \approx 7 \times 10^{-23}\mathrm{m}^{-2}$. This condition is much weaker than the mentioned cosmological upper bound.

For larger values the existence cannot be proved with the used arguments. Finiteness of the radius for a given equation of state is discussed, a necessary and a sufficient condition are shown.

Another aim is to derive an analogous Buchdahl inequality that includes the cosmological term. For positive cosmological constants the vacuum region of spacetime may contain a cosmological event horizon. With the analogous Buchdahl inequality it will be proved that stellar models have an upper bound given by that event horizon.

Collins [3] stated that for a fixed equation of state and cosmological constant the choice of central pressure and therefore central density does not uniquely determine the solution. This is disproved.

Static perfect fluid solutions with cosmological constant were analysed by Kriele [13] and later by Winter [28]. Both derived the analogous TOV-Λ equation. The first one shows uniqueness of the solution for given pressure

and density distributions, which already disproved Collins [3]. An analogous type of Buchdahl inequality is derived but not discussed in the context of upper and lower bounds on radii of stellar objects. Winter [28] integrates the TOV-Λ equation from the boundary inwards to the centre, without proving the existence of that boundary. This leads to solutions with non-regular centres and is therefore not suitable for discussing stellar models.

Constant density solutions with cosmological constant were first analysed by Weyl [27]. In this remarkable paper these solutions are described for different values of the cosmological constant. The different possible spatial geometries were already pointed out and a possible coordinate singularity was mentioned.

More than 80 years later Stuchlík [24] analysed these solutions again. He integrated the TOV-Λ equation for possible values of the cosmological constant up to the limit $\Lambda < 4\pi\rho_0$, where ρ_0 denotes this constant density. In these cases constant density solutions describe stellar models. The third chapter shows that a coordinate singularity occurs if $\Lambda \geq 4\pi\rho_0$, already mentioned by Weyl [27].

If the cosmological constant equals this upper bound the pressure vanishes at the mentioned coordinate singularity. In this case it is not possible to join the Schwarzschild-de Sitter solution for the vacuum. One has to use the Nariai solution [16, 17] to get the metric C^1 at the boundary. For larger cosmological constant the pressure will vanish after the coordinate singularity. The volume of group orbits is decreasing and there one has to join the Schwarzschild-de Sitter solution containing the $r = 0$ singularity. Increasing the cosmological constant further leads to generalisations of the Einstein static universe. These solutions have two regular centres with monotonically decreasing or increasing pressure from the first to the second centre. Certainly the Einstein cosmos itself is a solution. Another new kind describes solutions with a regular centre and increasing divergent pressure. In this case the spacetime has a geometrical singularity. These solutions are unphysical and therefore not of great interest.

With the arguments used in this diploma thesis the existence of a global solution with given equation of state can only be proved for cosmological constants satisfying the bound mentioned above, $\Lambda < 4\pi\rho_b$.

Unfortunately, with this restriction solutions of the new kinds do not occur. Thus one cannot prove existence of the new kinds of solutions for a prescribed equation of state.

In addition to what was said before this paper shortly investigates Newtonian limits of the Schwarzschild-de Sitter solution and of the TOV-Λ equation. In the first case both possible horizons shrink to a point in the limit.

In the second, as in the case without cosmological term, this leads to the Euler equation for hydrostatic equilibrium. Nonetheless both limits contain a Λ-corrected gravitational potential.

2 Cosmological TOV equation

This chapter deals with the Einstein field equations with cosmological constant in the spherically symmetric and static case. For a generalised Birkhoff theorem see [23, 20].

First a perfect fluid is assumed to be the matter source. This directly leads to a Λ-extended Tolman-Oppenheimer-Volkoff equation which will be called TOV-Λ equation. The TOV-Λ equation together with the mean density equation form a system of differential equations. It can easily be integrated if a constant density is assumed.

Next a vacuum solution, namely the Schwarzschild-de Sitter solution, is derived. It is the unique static, spherically symmetric vacuum solution to the field equations with cosmological constant with group orbits having non-constant volume. But there is one other static, spherically symmetric vacuum solution, the Nariai solution [16, 17]. The group orbits of this solution have constant volume. The Nariai solution is needed to join interior and exterior solution in a special case but will not be derived explicitly.

Finally in this introductory chapter Newtonian limits are derived. Limits of the Schwarzschild-de Sitter solution and the TOV-Λ equation are shown. The second leads to the fundamental equation of Newtonian astrophysics with cosmological constant.

2.1 The Cosmological Constant

Originally the Einstein field equations read

$$G_{\mu\nu} = \kappa T_{\mu\nu}, \tag{2.1}$$

where κ is the coupling constant. Later Einstein [7] introduced the cosmological constant mainly to get a static cosmological solution. These field equations are

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}. \tag{2.2}$$

The effect of Λ can be seen as a special type of an energy-momentum tensor. It acts as an unusual fluid

$$T^{\Lambda}_{\mu\nu} = -\frac{\Lambda}{\kappa} g_{\mu\nu} = (\rho^{\Lambda} + \frac{P^{\Lambda}}{c^2}) u_\mu u_\nu + P^{\Lambda} g_{\mu\nu},$$

with $P^{\Lambda} = -\Lambda/\kappa$ and equation of state $P^{\Lambda} = -\rho^{\Lambda} c^2$.

2.2 Remarks on the Newtonian limit

The metric for a static, spherically symmetric spacetime can be written

$$ds^2 = -c^2 e^{\nu(r)} dt^2 + e^{a(r)} dr^2 + r^2(d\theta^2 + \sin^2\theta \; d\phi^2). \tag{2.3}$$

Appendix B of [11] shows that r^2 in front of the sphere metric is no loss of generality for a perfect fluid. This is true for vanishing cosmological constant. With cosmological term there is one vacuum solution with group orbits of constant volume, the mentioned Nariai solution [16, 17].

Metric (2.3) contains the constant c representing the speed of light. Define $\lambda = 1/c^2$, roughly speaking the limit $\lambda \to 0$ corresponds to the Newtonian theory. But equations could be non-regular in the limit $\lambda = 0$. Therefore, following [4, 5, 6], the static, spherically symmetric metric has to be written

$$ds^2 = -\frac{1}{\lambda} e^{\lambda\nu(r)} dt^2 + e^{a(r)} dr^2 + r^2 d\Omega^2, \tag{2.4}$$

with field equations

$$G_{\mu\nu} + \lambda\Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{2.5}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta \; d\phi^2$. The new term $\lambda\nu(r)$ in (2.4) ensures regularity of the energy-momentum conservation in the limit if matter is present. The $\lambda\Lambda$ term in the field equations (2.5) makes sure that the gravitational potential is regular in the limit $\lambda = 0$.

2.3 TOV-Λ equation

Assume a perfect fluid to be the source of the gravitational field. The derivation of the analogous Tolman-Oppenheimer-Volkoff [18, 25] equation is shown. Consider metric (2.4)

$$ds^2 = -\frac{1}{\lambda} e^{\lambda\nu(r)} dt^2 + e^{a(r)} dr^2 + r^2 d\Omega^2.$$

Units where $G = 1$ will be used, thus $\kappa = 8\pi\lambda^2$. $\lambda = 1$ corresponds to Einstein's theory of gravitation in geometrised units.

The field equations (2.5) for a perfect fluid can be taken from appendix A, (A.4)-(A.7) with (A.9). These are three independent equations, which imply energy-momentum conservation (A.10). Thus one may either use three independent field equations or one uses two field equations and the energy-momentum conservation equation. The aim of this section is to derive a

system of differential equations for pressure and mean density, defined later. Hence it is more effective to do the second. So consider the first two field equations and the conservation equation. This gives

$$\frac{1}{\lambda r^2} e^{\lambda\nu(r)} \frac{d}{dr}\left(r - re^{-a(r)}\right) - \Lambda e^{\lambda\nu(r)} = 8\pi\rho(r)e^{\lambda\nu(r)} \tag{2.6}$$

$$\frac{1}{r^2}\left(1 + r\lambda\nu'(r) - e^{a(r)}\right) + \lambda\Lambda e^{a(r)} = 8\pi\lambda^2 P(r)e^{a(r)} \tag{2.7}$$

$$-\frac{\nu'(r)}{2}(\lambda P(r) + \rho) = P'(r). \tag{2.8}$$

These are three independent ordinary differential equations. But there are four unknown functions. Mathematically but not physically one of the four functions can be chosen freely. From a physical point of view there are two possibilities. Either a matter distribution $\rho = \rho(r)$ or an equation of state $\rho = \rho(P)$ is prescribed. The most physical case is to prescribe an equation of state. Equation (2.6) can easily be integrated. By putting the constant of integration equal to zero because of regularity at the centre one gets

$$e^{-a(r)} = 1 - \lambda 8\pi \frac{1}{r}\int_0^r s^2\rho(s)ds - \lambda\frac{\Lambda}{r}\int_0^r s^2 ds. \tag{2.9}$$

Now the definition of 'mass up to r' is used

$$m(r) = 4\pi\int_0^r s^2\rho(s)ds. \tag{2.10}$$

The second integral of (2.9) can be evaluated to finally give

$$e^{-a(r)} = 1 - \lambda\frac{2m(r)}{r} - \lambda\frac{\Lambda}{3}r^2. \tag{2.11}$$

In addition define the mean density up to r by

$$w(r) = \frac{m(r)}{r^3}, \tag{2.12}$$

then the above metric component takes the form

$$e^{-a(r)} = 1 - \lambda 2w(r)r^2 - \lambda\frac{\Lambda}{3}r^2. \tag{2.13}$$

Therefore one may write metric (2.4) as

$$ds^2 = -\frac{1}{\lambda}e^{\lambda\nu(r)}dt^2 + \frac{dr^2}{1 - \lambda 2w(r)r^2 - \lambda\frac{\Lambda}{3}r^2} + r^2 d\Omega^2. \tag{2.14}$$

The function $\nu'(r)$ can be eliminated from (2.7) and (2.8). The second field equation (2.7) yields to

$$\frac{\nu'(r)}{2} = r\frac{\lambda 4\pi P(r) + w(r) - \frac{\Lambda}{3}}{1 - \lambda 2w(r)r^2 - \lambda\frac{\Lambda}{3}r^2}. \tag{2.15}$$

This with (2.8) implies the TOV-Λ equation [24, 28]

$$P'(r) = -r\frac{\left(\lambda 4\pi P(r) + w(r) - \frac{\Lambda}{3}\right)\left(\lambda P(r) + \rho(r)\right)}{1 - \lambda 2w(r)r^2 - \lambda\frac{\Lambda}{3}r^2}. \tag{2.16}$$

Putting $\Lambda = 0$ in (2.16) one finds the Tolman-Oppenheimer-Volkoff equation, without cosmological term, short TOV equation, see [18, 25].

If an equation of state $\rho = \rho(P)$ is given, the conservation equation (2.8) can be integrated to give

$$\nu(r) = -\int_{P_c}^{P(r)} \frac{2dP}{\lambda P + \rho(P)}, \tag{2.17}$$

where P_c denotes the central pressure. Using the definition of $m(r)$ then (2.12) and (2.16) form an integro-differential system for $\rho(r)$ and $P(r)$. Differentiating (2.12) with respect to r implies

$$w'(r) = \frac{1}{r}\left(4\pi\rho(P(r)) - 3w(r)\right). \tag{2.18}$$

Therefore given $\rho = \rho(P)$ equations (2.16) and (2.18) are forming a system of differential equations in $P(r)$ and $w(r)$. The solution for $w(r)$ together with (2.13) gives the function $a(r)$, whereas the solution of $P(r)$ in (2.17) defines $\nu(r)$.

There is a alternative way of finding the TOV-Λ equation if the TOV equation is given. Looking back at the Einstein field equations with cosmological constant (2.6), (2.7) it is found that they may be derived by a simple substitution. Putting

$$\rho_{\text{eff}} = \rho + \frac{\Lambda}{8\pi},$$
$$P_{\text{eff}} = P - \frac{\Lambda}{\lambda 8\pi},$$

in the field equations without cosmological term (2.1), gives the Λ term properly. Therefore the effective energy-momentum tensor is defined by

$$T^{\text{eff}}_{\mu\nu} := T_{\mu\nu} - \frac{\Lambda}{\kappa}g_{\mu\nu}. \tag{2.19}$$

In addition it should be remarked how w_{eff} is given,

$$w_{\text{eff}} = w + \frac{\Lambda}{6}.$$

The possibility of writing the equations with effective values is of interest later.

2.4 Schwarzschild-anti-de Sitter and Schwarzschild-de Sitter solution

A vacuum solution ($T_{\mu\nu} = 0$) to the Einstein field equations with cosmological constant is derived. For a vanishing cosmological constant the Schwarzschild solution follows, for vanishing mass the metric gives the de Sitter cosmology. Again one uses metric (2.4)

$$ds^2 = -\frac{1}{\lambda} e^{\lambda\nu(r)} dt^2 + e^{a(r)} dr^2 + r^2 d\Omega^2.$$

One has to find a solution to the system of equations

$$G_{\mu\nu} + \lambda\Lambda g_{\mu\nu} = 0.$$

In the static, spherically symmetric case this system reduces to three equations. But there are only two unknown functions. The third equation is not independent. Thus it suffices to consider the first two field equations (A.4),(A.5), given in appendix A.

$$\frac{1}{\lambda r^2} e^{\lambda\nu(r)} \frac{d}{dr}\left(r - re^{-a(r)}\right) - \Lambda e^{\lambda\nu(r)} = 0 \tag{2.20}$$

$$\frac{1}{r^2}\left(1 + r\lambda\nu'(r) - e^{a(r)}\right) + \lambda\Lambda e^{a(r)} = 0. \tag{2.21}$$

The first equation gives

$$\frac{d}{dr}\left(r - re^{-a(r)}\right) - \lambda\Lambda r^2 = 0$$

and can be integrated to

$$e^{-a(r)} = 1 - \lambda\frac{2M}{r} - \lambda\frac{\Lambda}{3} r^2,$$

where $2M$ is a constant of integration. The additional λ in front of the constant of integration is to get interior (2.14) and exterior metric continuous at a boundary.

Again using equation (2.20), calculating the derivative and multiplying with $-e^{a(r)}$ one finds

$$-\left(e^{a(r)} + ra'(r) - 1\right) + \lambda\Lambda r^2 e^{a(r)} = 0.$$

The second field equation (2.21) can be put in the following form

$$\left(e^{a(r)} - r\lambda\nu'(r) - 1\right) - \lambda\Lambda r^2 e^{a(r)} = 0.$$

Adding up both equations gives $a'(r) + \lambda\nu'(r) = 0$. This can now be expressed with logarithms to give

$$\frac{d}{dr}\ln\left(e^{a(r)}e^{\lambda\nu(r)}\right) = 0.$$

One integrates this equation and may set the constant of integration equal to one because the time coordinate can be rescaled. This leads to

$$e^{a(r)}e^{\lambda\nu(r)} = 1,$$

and implies the static, spherically symmetric vacuum solution to the field equations with cosmological constant, see [23]. It is the unique solution which group orbits do not have constant volume.

With the above the following metric is obtained

$$ds^2 = -\frac{1}{\lambda}\left(1 - \lambda\frac{2M}{r} - \lambda\frac{\Lambda}{3}r^2\right)dt^2 + \frac{dr^2}{1 - \lambda\frac{2M}{r} - \lambda\frac{\Lambda}{3}r^2} + r^2 d\Omega^2. \quad (2.22)$$

Metric (2.22) is usually called Schwarzschild-de Sitter metric, although it was first published by Kottler (1918). It is well defined for radii satisfying $g^{rr}(r) > 0$.

Figure 28 shows Penrose-Carter diagram for the Schwarzschild-de Sitter space with $\lambda = 1$. The constants M and Λ are chosen such that $9\Lambda M^2 < 1$, which means that there are two horizons, see [10, 9]. If $9\Lambda M^2 = 1$ then there is only one horizon and if $9\Lambda M^2 > 1$ the spacetime does not contain a horizon.

Figure 29 shows Penrose-Carter diagram for Schwarzschild-anti-de Sitter space with $\lambda = 1$. For positive mass there is always the black-hole event horizon.

Setting $\Lambda = 0$ in (2.22) gives the usual Schwarzschild metric. For a vanishing mass the de Sitter metric is obtained.

2.5 Newtonian limits

The Newtonian limits of the Schwarzschild-anti-de Sitter and Schwarzschild-de Sitter metric and of the TOV-Λ equation are derived. The frame theory approach by Ehlers [4, 5, 6] will be used to define a one-parameter family of general relativistic spacetime models with Newtonian limit. Including the cosmological term gives a Newtonian limit with a Λ-corrected gravitational potential.

2.5.1 Limit of Schwarzschild-anti-de Sitter and Schwarzschild-de Sitter models

A parametrisation of the Schwarzschild-de Sitter metric is given by (2.22)

$$ds^2 = -\frac{1}{\lambda}\left(1 - \lambda\frac{2M}{r} - \lambda\frac{\Lambda}{3}r^2\right)dt^2 + \frac{dr^2}{1 - \lambda\frac{2M}{r} - \lambda\frac{\Lambda}{3}r^2} + r^2 d\Omega^2.$$

In the limit $\lambda = 0$ the family $\mathcal{M}(\lambda) = \left\{-\lambda g_{\alpha\beta}(\lambda), g^{\alpha\beta}(\lambda), \Gamma^{\gamma}_{\alpha\beta}(\lambda)\right\}$ converges to a field of a mass point at the origin. The black-hole event horizon and the cosmological event horizon both shrink to a point. The gravitational field is given by

$$\Gamma^{r}_{tt}[\lambda = 0] = \frac{M}{r^2} - \frac{\Lambda}{3}r, \tag{2.23}$$

all others vanish. This corresponds to Newton's equation with cosmological term

$$\nabla\phi(r) = \frac{M}{r^2} - \frac{\Lambda}{3}r.$$

2.5.2 Limit of the TOV-Λ equation

The parametrised metric (2.14) is given by

$$ds^2 = -\frac{1}{\lambda}e^{\lambda\nu(r)}dt^2 + \frac{dr^2}{1 - \lambda 2w(r)r^2 - \lambda\frac{\Lambda}{3}r^2} + r^2 d\Omega^2.$$

The parametrisation of the TOV-Λ equation (2.16) is

$$P'(r) = -r\frac{\left(4\pi\lambda P(r) + w(r) - \frac{\Lambda}{3}\right)\left(\lambda P(r) + \rho(r)\right)}{1 - 2\lambda w(r)r^2 - \lambda\frac{\Lambda}{3}r^2}.$$

This differential equation depends on λ, it is regular in the limit $\lambda = 0$. Solutions lead to a family of models with Newtonian limit.

The parametrised TOV-Λ equation leads to the "fundamental equation of Newtonian astrophysics" [26] with cosmological term

$$-r^2 P'(r)[\lambda = 0] = \rho(r) \left(m(r) - \frac{\Lambda}{3} r^3 \right).$$

The gravitational field reads

$$\Gamma^r_{tt}[\lambda = 0] = \frac{\nu'(r)}{2}[\lambda = 0], \tag{2.24}$$

all others vanish. Using (2.15) in (2.24) reproduces (2.23). But with the energy-momentum conservation (2.8) it gives

$$\Gamma^r_{tt}[\lambda = 0] = -\frac{P'(r)}{\rho(r)}, \tag{2.25}$$

which is the Euler equation for hydrostatic equilibrium in spherical symmetry, usually written as

$$\rho(r)\vec{f} = \nabla P(r).$$

3 Solutions with constant density

For practical reasons the notation is changed to geometrised units where $c^2 = 1/\lambda = 1$.

Assume a positive constant density distribution $\rho = \rho_0 = \text{const.}$ Then $w = \frac{4\pi}{3}\rho_0$ gives (2.16) in the form

$$P'(r) = -r\frac{\left(4\pi P(r) + \frac{4\pi}{3}\rho_0 - \frac{\Lambda}{3}\right)\left(P(r) + \rho_0\right)}{1 - \left(\frac{8\pi}{3}\rho_0 + \frac{\Lambda}{3}\right) r^2}. \tag{3.1}$$

In the following all solutions to the above differential equation are derived, see [21, 24, 25]. In [21, 25] the equation was integrated but not discussed for the different possible values of Λ. In [24] all cases with $\Lambda < 4\pi\rho_0$ were discussed using dimensionless variables. This complicated the physical interpretation of the results and was an unnecessary restriction to the cosmological constant.

The central pressure $P_c = P(r = 0)$ is always assumed to be positive and finite. Using that the density is constant in (2.17) metric (2.14) reads

$$ds^2 = -\left(\frac{P_c + \rho_0}{P(r) + \rho_0}\right)^2 dt^2 + \frac{dr^2}{1 - \left(\frac{8\pi}{3}\rho_0 + \frac{\Lambda}{3}\right) r^2} + r^2 d\Omega^2. \tag{3.2}$$

The metric is well defined for radial coordinates $r \in [0, \hat{r})$ if $\Lambda > -8\pi\rho_0$, where $\hat{r}$ denotes the zero of g^{rr}. If $\Lambda \leq -8\pi\rho_0$ the metric is well defined for all r.

Solutions of differential equation (3.1) are uniquely determined by the three parameters ρ_0, P_c and Λ. Therefore one has a 3-parameter family of solutions.

The two cases of negative and positive cosmological constant are discussed separately. In the first case there is no cosmological event horizon whereas in the second there is. Analogous Buchdahl [2] inequalities (3.38) will be derived for the different cases.

Figures of the pressure are shown. The constant density ρ_0 and the central pressure P_c are equal in all figures, $\rho_0 = P_c = 1$. Only the cosmological constant varies. This follows the approach of the chapter and helps to compare the different functions.

The end of the chapter summarises the different solutions. An overview is given.

3.1 Spatial geometry of solutions

Before starting to solve the TOV-Λ equation one should have a closer look at metric (3.2).

The aim of this section is to describe the spatial geometry of metric (3.2) because it depends on the choice of the constant density ρ_0 and the cosmological constant Λ. For vanishing cosmological constant this was done in [22].

For spherically symmetric spacetimes there exists an invariantly defined mass function [29], the quasilocal mass.

For metric (3.2) quasilocal mass [29] is given by

$$2m_{\mathrm{q}}(r) = \frac{8\pi}{3}\rho_0 r^3 + \frac{\Lambda}{3}r^3. \tag{3.3}$$

If R denotes the boundary of a stellar object then

$$2m_{\mathrm{q}}(R) = 2M + \frac{\Lambda}{3}R^3,$$

where M denotes the total mass of the object. Let

$$k = \frac{8\pi}{3}\rho_0 + \frac{\Lambda}{3},$$

then the the spatial part of metric (3.2) reads

$$d\sigma^2 = \frac{dr^2}{1-kr^2} + r^2 d\Omega^2, \tag{3.4}$$

and the quasilocal mass (3.3) is given by

$$2m_{\mathrm{q}}(r) = kr^3, \tag{3.5}$$

For all positive values of k spatial metric (3.4) describes one half of a 3-sphere of radius $1/\sqrt{k}$. The quasilocal mass $m_{\mathrm{q}}(r)$ is positive. Introducing a new coordinate by

$$r = \frac{1}{\sqrt{k}}\sin\alpha, \tag{3.6}$$

gives

$$d\sigma^2 = \frac{1}{k}(d\alpha^2 + \sin^2\alpha\ d\Omega^2).$$

This shows that (3.2) has a coordinate singularity. The metric with radial coordinate r only describes one half of the 3-sphere. Geometrically the coordinate singularity is easily explained. The volume of group orbits $r = \text{const.}$ has an extremum at the equator $\hat{r} = 1/\sqrt{k}$. Thus this coordinate cannot be used to describe the second half of the 3-sphere.

Differential equation (3.1) is singular at $\hat{r} = 1/\sqrt{k}$. With the new coordinate α the differential equation is regular at the the corresponding $\alpha(\hat{r}) = \pi/2$.

If $k = 0$ then $m_{\mathrm{q}}(r) = 0$ and the spatial metric is purely Euclidean in spherical coordinates. In Cartesian coordinates this simply reads

$$d\sigma^2 = dx^2 + dy^2 + dz^2.$$

Finally, negative values of k in the spacial metric correspond to three-dimensional hyperboloids with negative quasilocal mass. Then using

$$r = \frac{1}{\sqrt{-k}} \sinh \alpha, \tag{3.7}$$

gives

$$d\sigma^2 = \frac{1}{(-k)}(d\alpha^2 + \sinh^2\!\alpha \; d\Omega^2).$$

The coordinate singularity becomes important if Λ exceeds an upper limit. The restriction to $\Lambda < 4\pi\rho_0$ in [24] is due to this fact. Up to this limit the radial coordinate r and the new coordinate α are used side by side. Although the radial coordinate is less elegant the physical picture is clearer.

In appendix B it will be shown that constant density solutions are conformally flat [21]. Conformally flat constant density solutions were analysed in [23] and were called generalised solutions of the interior Schwarzschild solution.

If the density is decreasing or increasing outwards solutions are not conformally flat.

It will also be seen that models with a real geometric singularity exists. They are physically not relevant because the pressure is divergent at the singularity.

3.2 Solutions with negative cosmological constant

In the Schwarzschild-anti-de Sitter spacetime there exists a black-hole event horizon. It will be shown that radii of stellar objects are always larger than this radius corresponding to the black-hole event horizon.

3.2.1 Stellar models with spatially hyperbolic geometry

$\Lambda < -8\pi\rho_0$

If $\Lambda < -8\pi\rho_0$ then k and $m_{\mathrm{q}}(r)$ are negative. With (3.7) the denominator of (3.1) can be written as $(\cosh^2\alpha)$ and the differential equation does not have a singularity.

The volume of group orbits has no extrema, thus metric (3.2) has no coordinate singularities and is well defined for all r.

Write the differential equation (3.1) as

$$d\left[\ln\frac{3P+\rho_0-\frac{\Lambda}{4\pi}}{P+\rho_0}\right] = d\left[\ln\cosh\alpha\right]. \tag{3.8}$$

Integration gives

$$\frac{3P(\alpha)+\rho_0-\frac{\Lambda}{4\pi}}{P(\alpha)+\rho_0} = C\cosh\alpha. \tag{3.9}$$

C is the constant of integration, evaluated by defining $P(\alpha=0)=P_c$ to be the central pressure. At the centre it is found that

$$C = \frac{3P_c+\rho_0-\frac{\Lambda}{4\pi}}{P_c+\rho_0}. \tag{3.10}$$

Throughout this calculation it is assumed that $8\pi\rho_0 < -\Lambda$. With $P_c > 0$ this implies that $C > 3$ and $(1-\Lambda/4\pi\rho_0) > 3$, which can easily be verified. The constant C will be used instead of its explicit expression given by (3.10). Then the pressure reads

$$P(\alpha) = \rho_0\frac{\left(1-\frac{\Lambda}{4\pi\rho_0}\right)-C\cosh\alpha}{C\cosh\alpha-3}. \tag{3.11}$$

The function $P(\alpha)$ is well defined and monotonically decreasing for all α.

The pressure converges to $-\rho_0$ as α or as the radius r tends to infinity. Thus a radius R, where the pressure vanishes, always exists. It is given by

$$R^2 = \frac{3}{|8\pi\rho_0+\Lambda|}\left\{\frac{1}{C^2}\left(1-\frac{\Lambda}{4\pi\rho_0}\right)^2-1\right\}. \tag{3.12}$$

Writing it in α, where α_b corresponds to R

$$\cosh\alpha_b = \frac{1}{C}\left(1-\frac{\Lambda}{4\pi\rho_0}\right), \tag{3.13}$$

shows that the new coordinate α simplifies the expressions . Therefore the radial coordinate will only be used to give a physical picture or if it simplifies the calculations.

From equation (3.13) one can deduce the inverse function. Then one has the central pressure as a function of α_b, $P_c = P_c(\alpha_b)$. To do this the explicit expression for C from (3.10) is needed. One finds

$$P_c = \rho_0 \frac{\left(1 - \frac{\Lambda}{4\pi\rho_0}\right)(\cosh\alpha_b - 1)}{\left(1 - \frac{\Lambda}{4\pi\rho_0}\right) - 3\cosh\alpha_b}. \tag{3.14}$$

The central pressure given by (3.14) should be finite. Therefore one obtains an analogue of the Buchdahl inequality. It reads

$$\cosh\alpha_b < \frac{1}{9}\left(1 - \frac{\Lambda}{4\pi\rho_0}\right). \tag{3.15}$$

Thus there exists an upper bound for α_b for given Λ and ρ_0. Use that

$$\sinh(\text{arccosh}(\alpha)) = \sqrt{\alpha^2 - 1},$$

then the corresponding radius R is given by

$$R^2 < \frac{\frac{1}{3}\left(4 - \frac{\Lambda}{4\pi\rho_0}\right)}{4\pi\rho_0}. \tag{3.16}$$

Since the cosmological constant is negative the right-hand side of (3.16) is well defined. Using the definition of mass $M = (4\pi/3)\rho_0 R^3$ one can rewrite (3.16) in terms of M, R and Λ. This leads to

$$3M < \frac{2}{3}R + R\sqrt{\frac{4}{9} - \frac{\Lambda}{3}R^2}. \tag{3.17}$$

This is the wanted analogue of the Buchdahl inequality (3.38).

The explicit derivation of the last equation is shown in section 4.6, where the generalised Buchdahl inequality is derived.

To emphasise that the cosmological constant is negative write (3.17)

$$3M < \frac{2}{3}R + R\sqrt{\frac{4}{9} + \frac{|\Lambda|}{3}R^2}.$$

At $r = R$, where the pressure vanishes, the Schwarzschild-anti-de Sitter solution (2.22) is joined. Joining interior and exterior solution is needed in

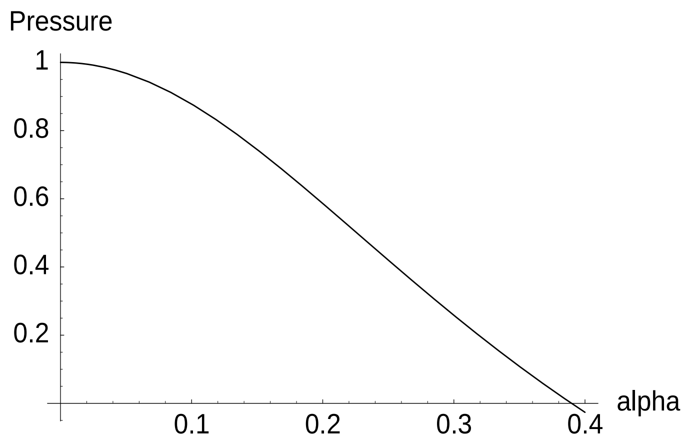

Figure 1: Pressure function $P(\alpha)$ with $\rho_0 = 1$, $P_c = 1$ and $\Lambda = -10\pi$

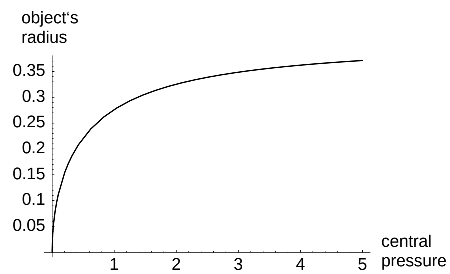

Figure 2: Radius as a function of central pressure $R = R(P_c)$

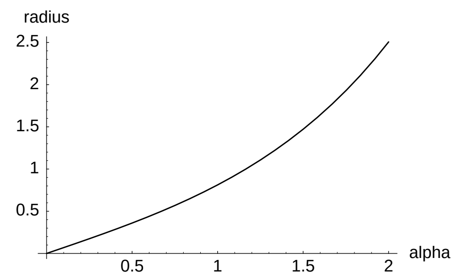

Figure 3: Radial coordinate $r = r(\alpha)$

this and many of the following cases. Therefore the next section shows this procedure explicitly.

Figure 1 shows the typical behaviour of function $P(\alpha)$ (3.11). Figures 2 and 3 show the radius of the object as a function of the central pressure (3.12) and the radius as a function of the new variable α (3.7), respectively. Constant density and central pressure are both set to one, $\rho_0 = 1$, $P_c = 1$.

3.2.2 Joining interior and exterior solution

At the $P = 0$ surface the Schwarzschild-anti-de Sitter solution (2.22) is joined. Since this is needed for all solutions describing stellar objects the procedure of joining interior and exterior solution is shown explicitly.

Gauss coordinates to the $r =$ const. hypersurfaces in the interior are

given by $\chi = \alpha/\sqrt{k}$. Then the interior metric (3.2) reads

$$ds^2 = -\left(\frac{P_c + \rho_0}{P(\chi) + \rho_0}\right)^2 dt^2 + d\chi^2 + \left(\frac{\sinh\sqrt{-k}\chi}{\sqrt{-k}}\right)^2 d\Omega^2.$$

The exterior metric, or the vacuum part of the solution, is given by (2.22). Gauss coordinates to the $r=$ const. hypersurfaces are defined by

$$d\chi = \frac{dr}{\sqrt{1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2}}.$$

The functions $\chi = \chi(r)$ and $r = r(\chi)$ are not given explicitly because they involve elliptic functions. However, one does not need the explicit form to show that the metric is C^1 at χ_b, where the pressure vanishes. In Gauss coordinates the exterior metric becomes

$$ds^2 = -\left(1 - \frac{2M}{r(\chi)} - \frac{\Lambda}{3}r(\chi)^2\right) dt^2 + d\chi^2 + r(\chi)^2 d\Omega^2. \tag{3.18}$$

Both metrics are joined at $P(\chi = \chi_b) = 0$. Continuity of the metric at χ_b implies that

$$\frac{\sinh\sqrt{-k}\chi_b}{\sqrt{-k}} = r(\chi_b). \tag{3.19}$$

To get the g_{tt}-component continuous one has to rescale the time in the interior. With

$$t \to \frac{\left(1 - \frac{2M}{r(\chi_b)} - \frac{\Lambda}{3}r(\chi_b)^2\right)}{\left(\frac{P_c+\rho_0}{\rho_0}\right)^2} t,$$

both metrics are joined continuously at χ_b. Continuity of the first derivatives imply

$$\cosh\sqrt{-k}\chi_b = \frac{dr}{d\chi}(\chi_b) = \sqrt{1 - \frac{2M}{r(\chi_b)} - \frac{\Lambda}{3}r(\chi_b)}. \tag{3.20}$$

If one shows that the derivatives of the g_{tt}-components are continuous at χ_b one is done. The interior part implies

$$\frac{dg_{tt}^{\mathrm{int}}}{d\chi}(\chi_b) = 2\left(1 - \frac{2M}{r(\chi_b)} - \frac{\Lambda}{3}r(\chi_b)\right)\frac{P'(\chi_b)}{\rho_0},$$

where $P'(\chi_b)$ can be derived from (3.1) to give

$$\frac{dP}{d\chi}(\chi_b) = -\frac{1}{\sqrt{-k}}\frac{\sinh\sqrt{-k}\chi_b}{\cosh\sqrt{-k}\chi_b}\left(\frac{4\pi}{3}\rho_0 - \frac{\Lambda}{3}\right)\rho_0.$$

With (3.20) both equations combine to

$$\frac{dg_{tt}^{\text{int}}}{d\chi}(\chi_b) = -2\frac{\sinh\sqrt{-k}\chi_b}{\sqrt{-k}}\cosh\sqrt{-k}\chi_b\left(\frac{4\pi}{3}\rho_0 - \frac{\Lambda}{3}\right). \tag{3.21}$$

For the exterior metric one finds

$$\frac{dg_{tt}^{\text{ext}}}{d\chi}(\chi_b) = -2r(\chi_b)\left(\frac{M}{r(\chi_b)^3} - \frac{\Lambda}{3}\right)\sqrt{1 - \frac{2M}{r(\chi_b)} - \frac{\Lambda}{3}r(\chi_b)}.$$

Use (3.19) for the first term and (3.20) for the last term. Furthermore the mass is given by

$$M = \frac{4\pi}{3}\rho_0\left(\frac{\sinh\sqrt{-k}\chi_b}{\sqrt{-k}}\right)^3 = \frac{4\pi}{3}\rho_0 r(\chi_b)^3,$$

see (2.10). Then this derivative evaluated at χ_b reads

$$\frac{dg_{tt}^{\text{ext}}}{d\chi}(\chi_b) = -2\frac{\sinh\sqrt{-k}\chi_b}{\sqrt{-k}}\cosh\sqrt{-k}\chi_b\left(\frac{4\pi}{3}\rho_0 - \frac{\Lambda}{3}\right), \tag{3.22}$$

which equals equation (3.21). Thus the metric is C^1 at χ_b.

Since the density is not continuous at the boundary the Ricci tensor is not, either. Therefore the metric is at most C^1. This cannot be improved.

3.2.3 Stellar models with spatially Euclidean geometry

$\Lambda = -8\pi\rho_0$

Assume that cosmological constant and constant density are chosen such that $8\pi\rho_0 = -\Lambda$. Then $k = 0$, $m_{\mathrm{q}}(r) = 0$ and the denominator of (3.1) becomes one. The differential equation simplifies to

$$\frac{dP}{dr} = -4\pi r\left(P + \rho_0\right)^2, \tag{3.23}$$

and the $t = $ const. hypersurfaces of (3.2) are purely Euclidean. As in the former case metric (3.2) is well defined for all r.

Equation (3.23) can be integrated. If the constant of integration is fixed at the centre by $P(r=0) = P_c$ one obtains

$$P(r) = \frac{1}{2\pi r^2 + \frac{1}{P_c+\rho_0}} - \rho_0. \tag{3.24}$$

The denominator of the pressure distribution cannot vanish because central pressure and density are assumed to be positive. Therefore (3.24) has no singularities.

As the radius tends to infinity the fraction tends to zero and thus the pressure converges to $-\rho_0$.

This implies that there always exits a radius R where $P(r = R) = 0$. Therefore all solutions to (3.23) in (3.2) describe stellar objects. Their radius is given by

$$R^2 = \frac{1}{2\pi}\left(\frac{1}{\rho_0} - \frac{1}{P_c+\rho_0}\right). \tag{3.25}$$

One finds that the radius R is bounded by $1/\sqrt{2\pi\rho_0}$. Inserting the definition of the density yields to

$$R^2 < \frac{1}{2\pi\rho_0} = \frac{1}{2\pi}\frac{4\pi R^3}{3M}, \tag{3.26}$$

which implies

$$3M < 2R. \tag{3.27}$$

Thus (3.27) is the analogous Buchdahl inequality to equation (3.38) and equals (3.17) with $\Lambda = -8\pi\rho_0 = -6M/R^3$.

At $r = R$, where the pressure vanishes, the Schwarzschild-anti-de Sitter solution (2.22) is joined uniquely by the same procedure as before. Since the density is not continuous at the boundary the Ricci tensor is not, either. Thus the metric is again at most C^1.

Figure 4 shows pressure (3.24) with spatially Euclidean geometry. The radius R where the pressure vanishes as a function of central pressure P_c (3.25) is show in figure 5. As before $\rho_0 = 1$, $P_c = 1$.

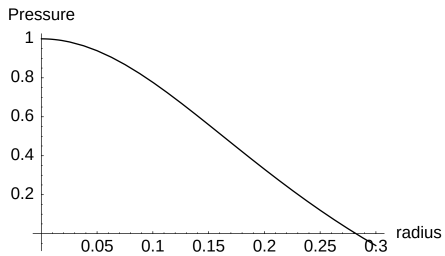

Figure 4: Pressure function $P(r)$ with $\rho_0 = 1$, $P_c = 1$ and $\Lambda = -8\pi$

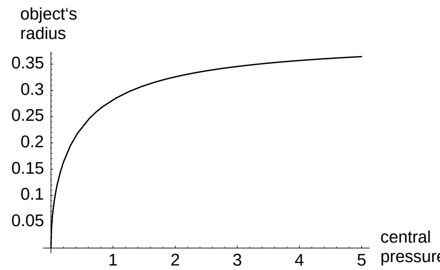

Figure 5: Radius as a function of central pressure $R = R(P_c)$

3.2.4 Stellar models with spatially spherical geometry

$\Lambda > -8\pi\rho_0$

If $\Lambda > -8\pi\rho_0$ then k and $m_{\mathrm{q}}(r)$ are positive. Equation (3.1) has a singularity at

$$r = \hat{r} = \frac{1}{\sqrt{\frac{8\pi}{3}\rho_0 + \frac{\Lambda}{3}}}. \tag{3.28}$$

In α the singularity $\hat{r}$ corresponds to $\alpha(\hat{r}) = \pi/2$.

As already mentioned in section 3.1 the spatial part of metric (3.2) now describes part of a 3-sphere of radius $1/\sqrt{k}$. The metric is well defined for radii less than $\hat{r}$.

Integration of (3.1) gives

$$P(\alpha) = \rho_0 \frac{\left(\frac{\Lambda}{4\pi\rho_0} - 1\right) + C\cos\alpha}{3 - C\cos\alpha}. \tag{3.29}$$

One finds

$$P(\pi/2) = \frac{\rho_0}{3}\left(\frac{\Lambda}{4\pi\rho_0} - 1\right), \tag{3.30}$$

which is less than zero if $\Lambda < 4\pi\rho_0$, which is the restriction in [24]. Since this is the considered case the singularity of the pressure gradient is not important yet.

C is a constant of integration and can be expressed by the central pressure P_c. Again one obtains

$$C = \frac{3P_c + \rho_0 - \frac{\Lambda}{4\pi}}{P_c + \rho_0}. \tag{3.31}$$

It is the same constant of integration as in (3.10). But possible value for C are different. The considered values of the cosmological constant imply that $1 < C < 3$. The pressure is decreasing for all α.

It was already stated that $P(\pi/2) < 0$. Thus there exists an α_b such that $P(\alpha_b) = 0$. α_b can be derived from equation (3.29) and one obtains the analogue of (3.12)

$$\cos\alpha_b = \frac{1}{C}\left(1 - \frac{\Lambda}{4\pi\rho_0}\right). \tag{3.32}$$

It remains to derive the analogue of the Buchdahl inequality for this case. One uses (3.32) to find $P_c = P_c(\alpha_b)$

$$P_c = \rho_0 \frac{\left(1 - \frac{\Lambda}{4\pi\rho_0}\right)(1 - \cos\alpha_b)}{3\cos\alpha_b - \left(1 - \frac{\Lambda}{4\pi\rho_0}\right)}, \tag{3.33}$$

which is similar to (3.12). Finiteness of the central pressure in (3.33) gives

$$\cos\alpha_b > \frac{1}{3}\left(1 - \frac{\Lambda}{4\pi\rho_0}\right). \tag{3.34}$$

With

$$\sin(\arccos(\alpha)) = \sqrt{1 - \alpha^2},$$

this yields to the wanted analogue

$$R^2 < \frac{\frac{1}{3}\left(4 - \frac{\Lambda}{4\pi\rho_0}\right)}{4\pi\rho_0}, \tag{3.35}$$

which equals (3.16). The greater sign reversed because $(\arccos\alpha)$ is decreasing. Thus it can also be rewritten to give

$$3M < \frac{2}{3}R + R\sqrt{\frac{4}{9} - \frac{\Lambda}{3}R^2}, \tag{3.36}$$

which equals (3.17). To emphasise that the cosmological constant is negative one may write $+|\Lambda|$ rather than $-\Lambda$.

At $\alpha = \alpha_b$ or $r = R$ the Schwarzschild-anti-de Sitter solution is joined and the metric is C^1. Without further assumptions this cannot be improved.

The pressure (3.29) is shown in figure 6. Figures 7 and 8 show the radius of the object as a function of the central pressure (3.32) and the radius as a function of the new variable α (3.6), respectively.

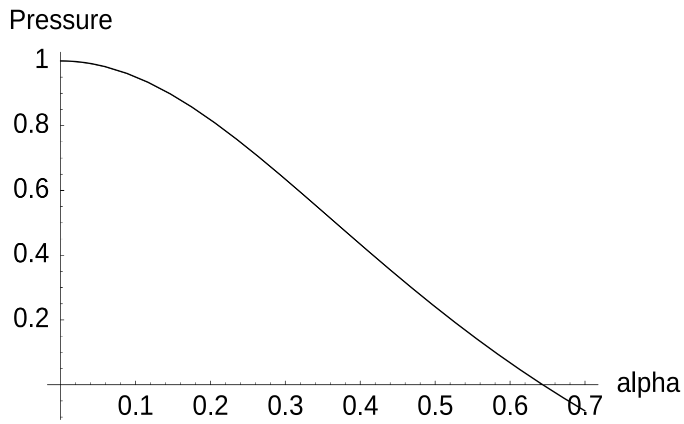

Figure 6: Pressure function $P(\alpha)$ with $\rho_0 = 1$, $P_c = 1$ and $\Lambda = -4\pi$

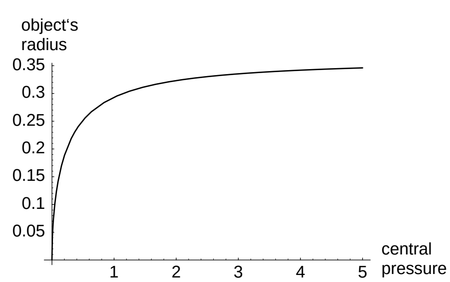

Figure 7: Radius as a function of central pressure $R = R(P_c)$

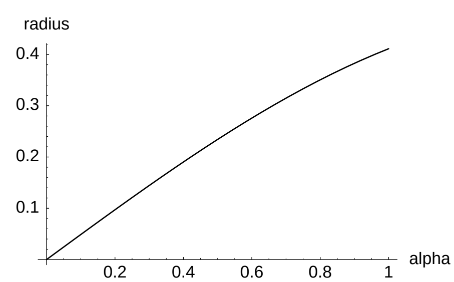

Figure 8: Radial coordinate $r = r(\alpha)$

3.3 Solutions with vanishing cosmological constant

Assume a vanishing cosmological constant. Then one can use all equations of the former case with $\Lambda = 0$. Only one of these relations will be shown, namely the Buchdahl inequality [2]. It is derived from (3.35)

$$R^2 < \frac{1}{3\pi\rho_0}, \tag{3.37}$$

using $M = (4\pi/3)\rho_0 R^3$ leads to

$$M < \frac{4}{9}R. \tag{3.38}$$

Because of the analogous inequalities of the former cases write

$$3M < \frac{4}{3}R. \tag{3.39}$$

For vanishing cosmological constant the pressure is shown in figure 9. Figures 10 and 11 show the radius of the object as a function of the central pressure and the radius as a function of the new variable α (3.6), respectively. The plotted pressure and central pressure are given by (3.29) and (3.32) with $\Lambda = 0$.

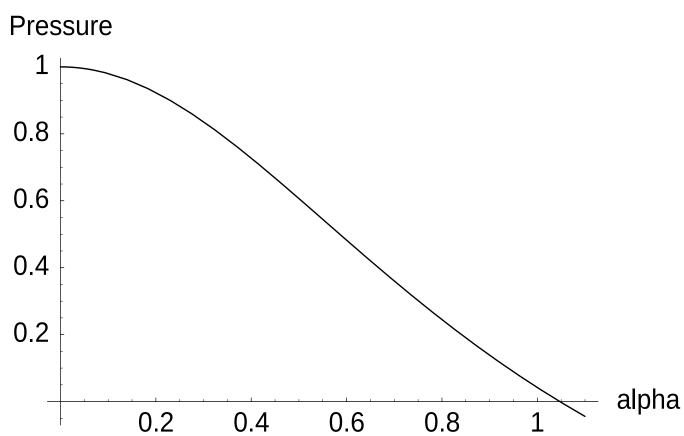

Figure 9: Pressure function $P(\alpha)$ with $\rho_0 = 1$, $P_c = 1$ and $\Lambda = 0$

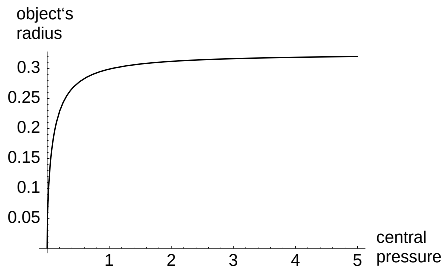

Figure 10: Radius as a function of central pressure $R = R(P_c)$

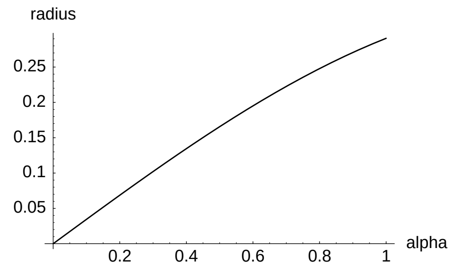

Figure 11: Radial coordinate $r = r(\alpha)$

3.4 Solutions with positive cosmological constant

In the Schwarzschild-de Sitter spacetime there may exist a black-hole event horizon and there may also exist a cosmological event horizon. It depends on M and Λ in the Schwarzschild-de Sitter metric (2.22) which of the cases occur. The three possible cases were mentioned in the end of section 2.4.

3.4.1 Stellar models with spatially spherical geometry

$0 < \Lambda < 4\pi\rho_0$

Integration of the TOV-Λ equation (3.1) leads to

$$P(\alpha) = \rho_0 \frac{\left(\frac{\Lambda}{4\pi\rho_0} - 1\right) + C\cos\alpha}{3 - C\cos\alpha}. \tag{3.40}$$

Because of (3.30) the boundary $P(R) = 0$ exists. As in section 3.2.4 one finds

$$R^2 < \frac{\frac{1}{3}\left(4 - \frac{\Lambda}{4\pi\rho_0}\right)}{4\pi\rho_0}, \tag{3.41}$$

and written in terms of M, R and Λ again gives

$$3M < \frac{2}{3}R + R\sqrt{\frac{4}{9} - \frac{\Lambda}{3}R^2}. \tag{3.42}$$

Since the cosmological constant is positive the square root term is well defined if

$$R \leq \sqrt{\frac{4}{3}}\frac{1}{\sqrt{\Lambda}}. \tag{3.43}$$

In this section $0 < \Lambda < 4\pi\rho_0$ is assumed. Using the definition of mass this can be rewritten to give

$$\Lambda < 4\pi\rho_0 = \frac{3M}{R^3}.$$

This leads to

$$9\Lambda M^2 < \left(\frac{3M}{R}\right)^3, \tag{3.44}$$

where the right-hand side of (3.44) has an upper bound given by (3.42). If in addition $R < 1/\sqrt{\Lambda}$ then the right-hand side of (3.44) is bounded by one and the exterior Schwarzschild-de Sitter spacetime has two horizons, see section 2.4. Without the additional assumption the exterior spacetime may also have one or may also have no horizon.

The Schwarzschild-de Sitter solution is joined at $r = R$, where $P(R) = 0$. Defining $M = (4\pi/3)\rho_0 R^3$ and rescaling the time in the interior the metric will be continuous at R. Again Gauss coordinates relative to the hypersurface $P(R) = 0$ can be used to get the metric C^1 at R. Since the density is not continuous at the boundary this cannot be improved.

One can construct solutions without singularities. The group orbits are increasing up to R. The boundary of the stellar object $r = R$ can be put in region I of Penrose-Carter figure 28 where the time-like killing vector is future directed. This leads to figure 30, the spacetime still contains an infinite sequence of singularities $r = 0$ and space-like infinities $r = \infty$. It is possible to put a second object with boundary $r = R$ in region IV of Penrose-Carter figure 30 where the time-like killing vector is past directed. This leads to figure 31 and shows that there are no singularities in the spacetime. This spacetime is not globally static because of the remaining dynamical parts of the the Schwarzschild-de Sitter spacetime, regions II_C and III_C. Penrose-Carter diagrams are in appendix C.

Similarly to the last figures pressure (3.40), central pressure (3.32) and the new variable α (3.6) are shown in figure 12, figure 13 and figure 14, respectively.

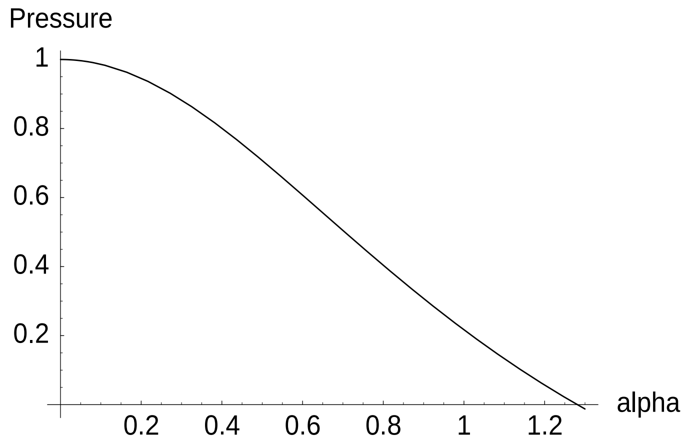

Figure 12: Pressure function $P(\alpha)$ with $\rho_0 = 1$, $P_c = 1$ and $\Lambda = 2\pi$

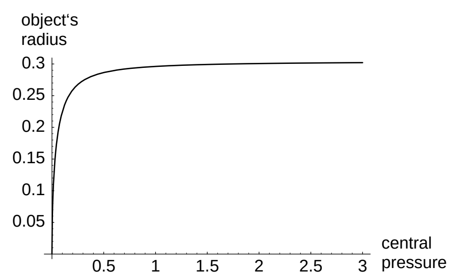

Figure 13: Radius as a function of central pressure $R = R(P_c)$

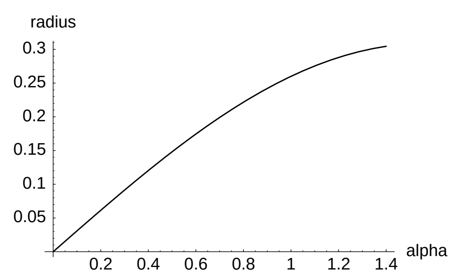

Figure 14: Radial coordinate $r = r(\alpha)$

3.4.2 Solutions with exterior Nariai metric

$\Lambda = 4\pi\rho_0$

In this special case integration of (3.1) gives

$$P(\alpha) = \rho_0 \frac{C\cos\alpha}{3 - C\cos\alpha}. \tag{3.45}$$

The pressure (3.45) vanishes at $\alpha = \alpha_b = \pi/2$, the coordinate singularity in the radial coordinate r.

One would like to join the Schwarzschild-de Sitter metric. But with $\Lambda = 4\pi\rho_0$ and $M = (4\pi/3)\rho_0 R^3$ it reads

$$ds^2 = -\left(1 - \frac{3M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{3M}{r}} + r^2 d\Omega^2. \tag{3.46}$$

The corresponding radius to α_b is $R = 3M$. Therefore one would join both solutions at the horizon $R = 3M$. Since $\Lambda = 4\pi\rho_0$ the radius R is also given by the cosmological constant $R = 1/\sqrt{\Lambda}$. Thus the black-hole event horizon $3M$ and the cosmological event horizon $1/\sqrt{\Lambda}$ are both located at R.

The interior metric reads

$$ds^2 = -\left(1 - \frac{P_c}{P_c + \rho_0}\cos\alpha\right)^2 \left(\frac{P_c + \rho_0}{\rho_0}\right)^2 dt^2 + \frac{1}{\Lambda}\left[d\alpha^2 + d\Omega^2\right]. \tag{3.47}$$

The volume of group orbits of metric (3.46) is increasing whereas the group orbits of (3.47) have constant volume. Therefore it is not possible to join the vacuum solution (3.46) on as an exterior field to get the metric C^1 at the $P = 0$ surface.

But there is the other spherically symmetric vacuum solution to the Einstein field equations with cosmological constant, the Nariai solution [16, 17], mentioned in chapter 2. Its metric is

$$ds^2 = \frac{1}{\Lambda}\left[-(A\cos\log r + B\sin\log r)^2 dt^2 + \frac{1}{r^2}dr^2 + d\Omega^2\right], \tag{3.48}$$

where A and B are arbitrary constants. With $r = e^\alpha$ this becomes

$$ds^2 = \frac{1}{\Lambda}\left[-(A\cos\alpha + B\sin\alpha)^2 dt^2 + d\alpha^2 + d\Omega^2\right]. \tag{3.49}$$

Metrics (3.47) and (3.49) can be joined by fixing the constants A and B. With

$$A = -\frac{P_c}{\rho_0}\sqrt{\Lambda}, \; B = \frac{P_c + \rho_0}{\rho_0}\sqrt{\Lambda},$$

the metric is C^1 at $\alpha = \pi/2$. As before, since the density is not continuous, the metric is at most C^1.

The three figures show $P(\alpha)$, $R = R(P_c)$ and $r = r(\alpha)$, respectively.

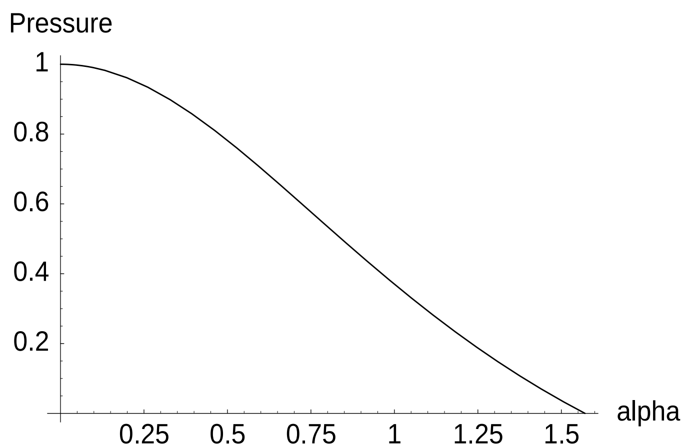

Figure 15: Pressure function $P(\alpha)$ with $\rho_0 = 1$, $P_c = 1$ and $\Lambda = 4\pi$

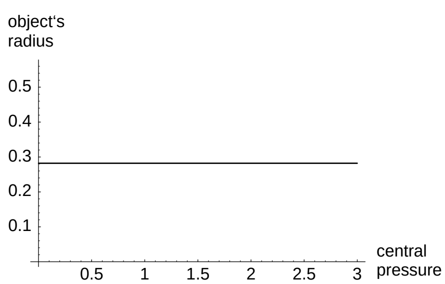

Figure 16: Radius as a function of central pressure $R = R(P_c)$

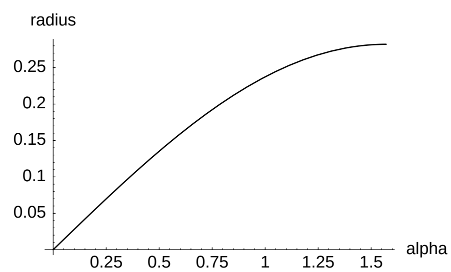

Figure 17: Radial coordinate $r = r(\alpha)$

3.4.3 Solutions with decreasing group orbits at the boundary

$4\pi\rho_0 < \Lambda < \Lambda_0$

Integration of (3.1) gives the pressure

$$P(\alpha) = \rho_0 \frac{\left(\frac{\Lambda}{4\pi\rho_0} - 1\right) + C\cos\alpha}{3 - C\cos\alpha}. \tag{3.50}$$

Assume that the pressure vanishes before the second centre $\alpha = \pi$ of the 3-sphere is reached. The condition $P(\alpha = \pi) < 0$ leads to an upper bound for the cosmological constant. This bound is given by

$$\Lambda_0 := 4\pi\rho_0 \left(\frac{4\,P_c/\rho_0 + 2}{P_c/\rho_0 + 2}\right). \tag{3.51}$$

Then $4\pi\rho_0 < \Lambda < \Lambda_0$ implies the following:

The pressure is decreasing near the centre and vanishes for some α_b, where $\pi/2 < \alpha_b < \pi$. Equations (3.4) and (3.6) imply that the volume of group orbits is decreasing if $\alpha > \pi/2$.

At α_b one uniquely joins the Schwarzschild-de Sitter solution by $M = (4\pi/3)\rho_0 R^3$. With Gauss coordinates relative to the $P(\alpha_b) = 0$ hypersurface the metric will be C^1. But there is a crucial difference to the former case with exterior Schwarzschild-de Sitter solution. Because of the decreasing group orbits at the boundary there is still the singularity $r = 0$ in the vacuum spacetime. Penrose-Carter diagram 32 shows this interesting solution.

Figure 18 shows pressure (3.50). Figures 19 and 20 show the radius of the object as a function of the central pressure and the radius as a function of the new variable α, respectively. Constant density and central pressure are both set to one, $\rho_0 = 1$, $P_c = 1$.

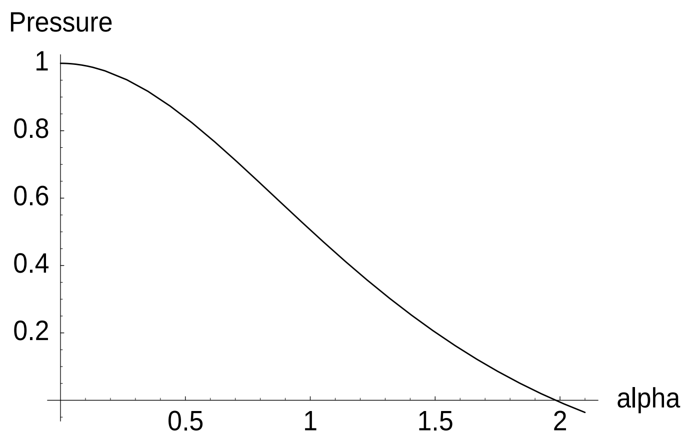

Figure 18: Pressure function $P(\alpha)$ with $\rho_0 = 1$, $P_c = 1$ and $\Lambda = 6\pi$

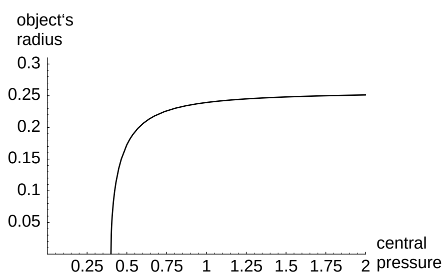

Figure 19: Radius as a function of central pressure $R = R(P_c)$

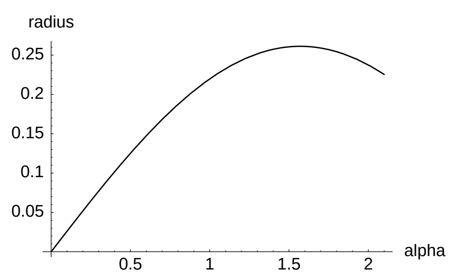

Figure 20: Radial coordinate $r = r(\alpha)$

3.4.4 Decreasing solutions with two regular centres

$\Lambda_0 \leq \Lambda < \Lambda_E$

As before the pressure is given by

$$P(\alpha) = \rho_0 \frac{\left(\frac{\Lambda}{4\pi\rho_0} - 1\right) + C\cos\alpha}{3 - C\cos\alpha}.$$

Assume that the pressure is decreasing near the first centre $\alpha = 0$. This gives an upper bound of the cosmological constant

$$\Lambda_E := 4\pi\rho_0 \left(3\, P_c/\rho_0 + 1\right), \tag{3.52}$$

where Λ_E is the cosmological constant of the Einstein static universe. These possible values $\Lambda_0 \leq \Lambda < \Lambda_E$ imply:

The pressure is decreasing is near the first centre $\alpha = 0$ but remains positive for all α because $\Lambda \geq \Lambda_0$. Therefore there exists a second centre at $\alpha = \pi$. At the second centre of the 3-sphere the pressure becomes

$$P(\alpha = \pi) = \rho_0 \frac{\left(\frac{\Lambda}{4\pi\rho_0} - 1\right) - C}{3 + C}. \tag{3.53}$$

It only vanishes if $\Lambda = \Lambda_0$. The solution is inextendible. The second centre is also regular. This is easily shown with Gauss coordinates. Recall metric (3.2) written in α

$$ds^2 = -\left(\frac{P_c + \rho_0}{P(\alpha) + \rho_0}\right)^2 dt^2 + \frac{1}{k}\left(d\alpha^2 + \sin^2\alpha d\Omega^2\right).$$

So one already used nearly Gauss coordinates up to a rescaling. With $\chi = \alpha/\sqrt{k}$ this becomes

$$ds^2 = -\left(\frac{P_c + \rho_0}{P(\chi) + \rho_0}\right)^2 dt^2 + d\chi^2 + \left(\frac{\sin\sqrt{k}\chi}{\sqrt{k}}\right)^2 d\Omega^2. \tag{3.54}$$

Thus the radius in terms of the Gauss coordinate is

$$r(\chi) = \frac{1}{\sqrt{k}}\sin(\sqrt{k}\chi), \tag{3.55}$$

and both centres are regular because

$$\frac{d}{d\chi} r(\chi) = \cos(\sqrt{k}\chi) = \pm 1 \text{ for } \sqrt{k}\chi = 0, \pi. \tag{3.56}$$

Solutions of this kind are generalisations of the Einstein static universe. These 3-spheres have a homogenous density but do not have constant pressure. They have a given central pressure P_c at the first regular centre which decreases monotonically towards the second regular centre. Generalisations of the Einstein static universe have been published earlier in Ref. [12].[1]

The pressure is plotted in figure 21 with $\rho_0 = 1$ and $P_c = 1$. Figure 22 shows the radius as a function of α.

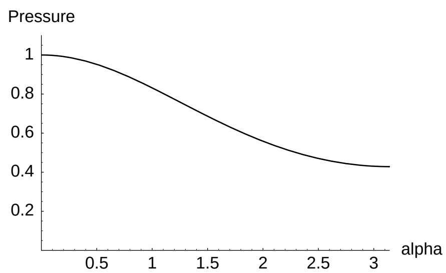

Figure 21: Pressure function $P(\alpha)$ with $\rho_0 = 1$, $P_c = 1$ and $\Lambda = 12\pi$

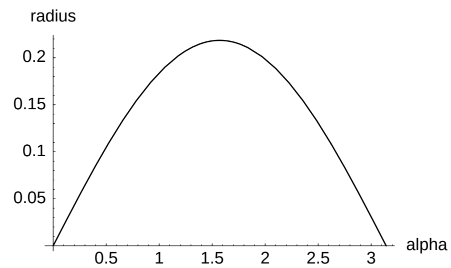

Figure 22: Radial coordinate $r = r(\alpha)$

[1] I would like to thank Aysel Karafistan for bringing this reference to my attention.

3.4.5 The Einstein static universe

$\Lambda = \Lambda_E$

Assume a constant pressure function. Then the pressure gradient vanishes and therefore the right-hand side of (3.1) has to vanish. This gives an unique relation between density, central pressure and cosmological constant. One obtains (3.52)

$$\Lambda = \Lambda_E = 4\pi \left(3P_E + \rho_0\right),$$

where $P_E = P_c$ was used to emphasise that the given central pressure corresponds to the Einstein static universe and is homogenous. Its metric is given by

$$ds^2 = -dt^2 + \mathcal{R}_E \left(d\alpha^2 + \sin^2\alpha d\Omega^2\right),$$

where

$$\mathcal{R}_E = \frac{1}{\sqrt{4\pi(P_E + \rho_E)}}.$$

$\mathcal{R}_E$ is the radius of the three dimensional hyper-sphere $t =$ const. The time component was rescaled to 1.

Thus for a given density ρ_0 there exists for every choice of a central pressure P_c a unique cosmological constant given by (3.52) such that an Einstein static universe is the solution to (3.1).

The constant pressure of the Einstein static universe is plotted in figure 23 with $\rho_0 = 1$ and $P_c = 1$. Figure 24 shows the radius as a function of α. Solutions with two centres all have the same radial coordinate $r = r(\alpha)$.

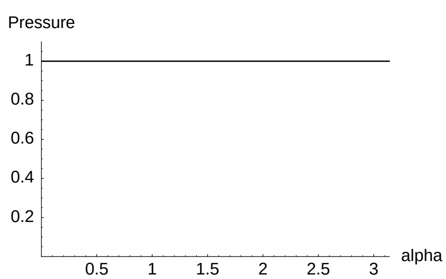

Figure 23: Pressure function $P(\alpha)$ with $\rho_0 = 1$, $P_c = 1$ and $\Lambda = 16\pi$

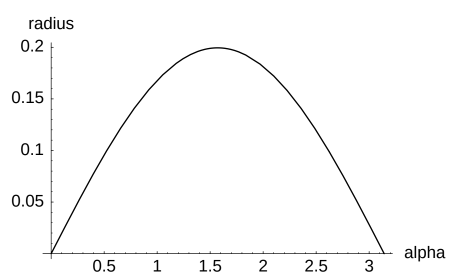

Figure 24: Radial coordinate $r = r(\alpha)$

3.4.6 Increasing solutions with two regular centres

$\Lambda_E < \Lambda < \Lambda_S$

Integration of (3.1) again leads to

$$P(\alpha) = \rho_0 \frac{\left(\frac{\Lambda}{4\pi\rho_0} - 1\right) + C\cos\alpha}{3 - C\cos\alpha}. \tag{3.57}$$

Assume that the pressure is finite at the second centre. This leads to an upper bound of the cosmological constant defined by

$$\Lambda_S := 4\pi\rho_0\left(6\,P_c/\rho_0 + 4\right). \tag{3.58}$$

The possible values of the cosmological constant imply:

The pressure $P(\alpha)$ is increasing near the first regular centre. It increases monotonically up to $\alpha = \pi$, where one has a second regular centre. This situation is similar to the case where $\Lambda_0 \leq \Lambda < \Lambda_E$. These solutions are also describing generalisations of the Einstein static universe.

More can be concluded. The generalisations are symmetric with respect to the Einstein static universe. By symmetric one means the following. Instead of writing the pressure as a function of α depending on the given values ρ_0, P_c and Λ one can eliminate the cosmological constant with the pressure at the second centre $P(\alpha = \pi)$, given by (3.53). Let P_{c1} and P_{c2} denote the pressures of the first and second centre, respectively. Then the pressure can be written as

$$P(\alpha)[P_{c1}, P_{c2}] = \rho_0 \frac{2P_{c1}P_{c2}\frac{1}{\rho_0} + (P_{c1} + P_{c2}) + (P_{c1} - P_{c2})\cos\alpha}{2\rho_0 + (P_{c1} + P_{c2}) - (P_{c1} - P_{c2})\cos\alpha}. \tag{3.59}$$

This implies

$$P(\frac{\pi}{2} + \alpha)[P_{c1}, P_{c2}] = P(\frac{\pi}{2} - \alpha)[P_{c2}, P_{c1}]. \tag{3.60}$$

Thus the pressure is symmetric to $\alpha = \pi/2$ if both central pressures are exchanged and therefore this is the converse situation to the case where $\Lambda_0 \leq \Lambda < \Lambda_E$.

Another way of looking at it is to consider the pressure in terms of effective values, mentioned in section 2.3. Then the pressure reads

$$P^{\text{eff}}(\alpha) = \rho_0^{\text{eff}} \frac{-1 + C^{\text{eff}}\cos\alpha}{3 - C^{\text{eff}}\cos\alpha}, \tag{3.61}$$

where

$$C^{\text{eff}} = \frac{3P_c^{\text{eff}} + \rho_0^{\text{eff}}}{P_c^{\text{eff}} + \rho_0^{\text{eff}}}. \tag{3.62}$$

The effective pressure at the second centre is given by $P_{c2}^{\text{eff}} = P^{\text{eff}}(\alpha = \pi)$. Writing P_{c1}^{eff} for the effective pressure at the fist centre and use (3.61) one can express ρ_0^{eff} by the two effective central pressures. Since this leads to an equation quadratic in ρ_0^{eff} there exist two effective constant densities for given effective central pressures. Therefore the decomposition of the effective energy-momentum tensor (2.19) in perfect fluid part and cosmological constant part is not unique.

If the central pressures are equal the dependence on α vanishes and one is left with the Einstein static universe.

The pressure of solutions with increasing pressure near the first centre is shown in figure 25. The radial coordinate is plotted in figure 26, see the last two cases.

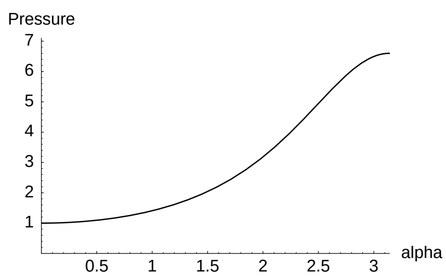

Figure 25: Pressure function $P(\alpha)$ with $\rho_0 = 1$, $P_c = 1$ and $\Lambda = 30\pi$

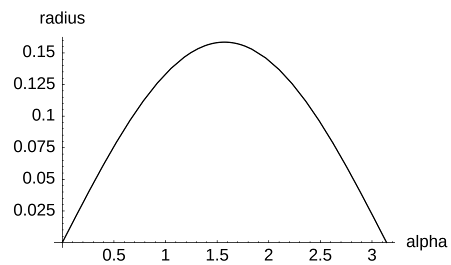

Figure 26: Radial coordinate $r = r(\alpha)$

3.4.7 Solutions with geometric singularity

$\Lambda \geq \Lambda_S$

In this case it is assumed that Λ exceeds the upper limit Λ_S. Then (3.57) implies that the pressure is increasing near the centre and diverges before $\alpha = \pi$ is reached. In appendix B it is shown that the divergence of the pressure implies divergence of the squared Riemann tensor (B.3). Thus these solutions have a geometric singularity with unphysical properties. Therefore they are of no further interest.

Figure 27 shows the divergent pressure.

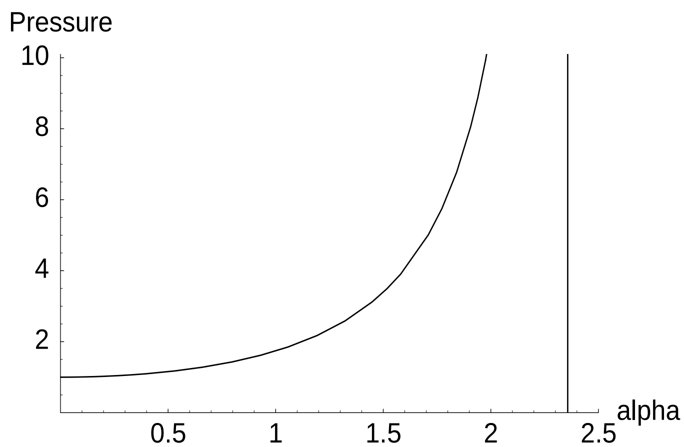

Figure 27: Pressure function $P(\alpha)$ with $\rho_0 = 1$, $P_c = 1$ and $\Lambda = 50\pi$

3.5 Overview of constant density solutions

cosmological constant	spatial geometry	short description of the solution
$\Lambda < -8\pi\rho_0$	hyperboloid	stellar model with exterior Schwarzschild-anti-de Sitter solution
$\Lambda = -8\pi\rho_0$	Euclidean	stellar model with exterior Schwarzschild-anti-de Sitter solution
$-8\pi\rho_0 < \Lambda < 0$	3–sphere	stellar model with exterior Schwarzschild-anti-de Sitter solution
$\Lambda = 0$	3–sphere	stellar model with exterior Schwarzschild solution
$0 < \Lambda < 4\pi\rho_0$	3–sphere	stellar model with exterior Schwarzschild-de Sitter solution
$\Lambda = 4\pi\rho_0$	3–sphere	stellar model with exterior Nariai solution
$4\pi\rho_0 < \Lambda < \Lambda_0$	3–sphere	decreasing pressure model with exterior Schwarzschild-de Sitter solution; the group orbits are decreasing at the boundary
$\Lambda_0 \leq \Lambda < \Lambda_E$	3–sphere	solution with two centres, pressure decreasing near the first; generalisation of the Einstein static universe
$\Lambda = \Lambda_E$	3–sphere	Einstein static universe
$\Lambda_E < \Lambda < \Lambda_S$	3–sphere	solution with two centres, solution increasing near the first; generalisation of the Einstein static universe
$\Lambda \geq \Lambda_S$	3–sphere	increasing pressure solution with geometric singularity

4 Solutions with given equation of state

This chapter analyses the system of differential equations for a given monotonic equation of state. The choice of central pressure and central density together with the cosmological constant uniquely determines the pressure. The chapter is based on [19], where the following analysis was done without cosmological term. Geometrised units where $c^2 = 1/\lambda = 1$ are used.

The existence of a global solution for cosmological constants satisfying $\Lambda < 4\pi\rho_b$ is shown, where ρ_b denotes the boundary density.

It turns out that solutions with negative cosmological constant are similar to those with vanishing cosmological constant. For positive cosmological constants new properties arise.

In section 4.1 Buchdahl variables are introduced. These new variables will help to proof the existence of global solutions and of stellar models. They are also helpful to derive analogous Buchdahl inequalities.

Section 4.2 shows the existence of a regular solution at the centre. After remarks on extending the solution and a possible coordinate singularity the existence of global solutions is shown in section 4.5.

For stellar models an analogous Buchdahl inequality is derived. Solutions without singularities are constructed. In particular for positive cosmological constants this is interesting because a second object has to be put in the spacetime to get a singularity free solution.

The chapter ends with some remarks on finiteness of the radius.

4.1 Buchdahl variables

In chapter 2 metric function $\nu(r)$ was eliminated in (2.7) and (2.8) to give the TOV-Λ equation. Alternatively the pressure can be eliminated in the same equations. To do this Buchdahl [2] originally introduced the following variables:

$$\begin{aligned} y^2 &= e^{-a(r)} = 1 - 2w(r)r^2 \\ \zeta &= e^{\nu/2} \\ x &= r^2. \end{aligned}$$

It seems to be natural to do exactly the same, but using the new expression for $e^{-a(r)}$ which involves the cosmological constant. Alternatively it can be

seen as $e^{-a(r)}$ written in effective values. The new Buchdahl variables are

$$y^2 = e^{-a(r)} = 1 - 2w(r)r^2 - \frac{\Lambda}{3}r^2 \tag{4.1}$$

$$\zeta = e^{\nu/2} \tag{4.2}$$

$$x = r^2. \tag{4.3}$$

The derivation of the equation one looks for is unpleasant. Anyway it is a straight forward calculation only involving basic algebra. Nonetheless the derivation is shown in detail because of the interesting result: The contribution of the cosmological constant is fully absorbed by the new Buchdahl variable y, no other terms involving Λ appear in the equation.

Start with the second field equation (2.7). Putting in the new variables leads to

$$y^2 + y^2\sqrt{x}\,2\sqrt{x}\,\nu_{,x} - 1 = (8\pi P - \Lambda)\,x.$$

With

$$\frac{y^2-1}{x} = -2w - \frac{\Lambda}{3},$$

this gives

$$8\pi P - \frac{2}{3}\Lambda = 4y^2\frac{\zeta_{,x}}{\zeta} - 2w, \tag{4.4}$$

where $\frac{d}{dr} = 2\sqrt{x}\frac{d}{dx}$ was used. The term $\frac{\zeta_{,x}}{\zeta}$ comes into the equation because $\nu_{,x} = 2\frac{\zeta_{,x}}{\zeta}$ which follows from its definition (4.2). Differentiating with respect to x implies

$$8\pi P_{,x} = 4\left(2yy_{,x}\frac{\zeta_{,x}}{\zeta} - y^2\frac{\zeta_{,xx}\zeta - \zeta_{,x}^2}{\zeta^2}\right) - 2w_{,x}. \tag{4.5}$$

Next the conservation equation (2.8) is rewritten. This then gives

$$P_{,x} = -\frac{\zeta_{,x}}{\zeta}(P+\rho) = -\frac{\zeta_{,x}}{\zeta}\left(P + \frac{1}{4\pi}(3w + 2xw_{,x})\right), \tag{4.6}$$

where (2.18) was used to eliminate ρ. Putting equations (4.4) and (4.5) in (4.6) eliminates the pressure. The equation in full details is

$$\frac{1}{2\pi}\left(2yy_{,x}\frac{\zeta_{,x}}{\zeta} - y^2\frac{\zeta_{,xx}\zeta - \zeta_{,x}^2}{\zeta^2}\right) - \frac{1}{4\pi}w_{,x} =$$
$$-\frac{\zeta_{,x}}{\zeta}\left(\frac{1}{8\pi}\left(4y^2\frac{\zeta_{,x}}{\zeta} + \frac{2}{3}\Lambda\right) - \frac{1}{4\pi}w + \frac{1}{4\pi}(2xw_{,x} + 3w)\right).$$

It is obvious that the terms with $y^2\zeta_{,x}^2$ drop out. Multiplying with ζ and using that

$$2yy_{,x} = -2w - 2xw_{,x} - \frac{\Lambda}{3}, \tag{4.7}$$

the following equation is obtained

$$y^2\zeta_{,xx} - \left(w + xw_{,x} + \frac{\Lambda}{6}\right)\zeta_{,x} - \frac{1}{2}w_{,x}\zeta = 0. \tag{4.8}$$

Using again (4.7) one finds that

$$y^2\zeta_{,xx} + yy_{,x}\zeta_{,x} - \frac{1}{2}w_{,x}\zeta = 0, \tag{4.9}$$

or rewritten using the product rule

$$(y\zeta_{,x})_{,x} - \frac{1}{2}\frac{w_{,x}\zeta}{y} = 0. \tag{4.10}$$

The last two differential equations are identical to these obtained without a cosmological constant and as mentioned above, Λ is fully absorbed by the new variable y. The equation is linear in ζ and w.

Derivation of (4.10) can also be done by rewriting the equation with vanishing cosmological constant in effective values for pressure and density. This leads to the Buchdahl variable y with w_{eff}.

4.2 Existence of a unique regular solution at the centre

It is shown that there exists a unique regular solution in a neighbourhood of the centre $r = 0$ for each central density and given equation of state. To do this the following theorem is needed, see [19] for a proof.

Theorem 1 *Let V be a finite dimensional real vector space, $N : V \to V$ a linear mapping, $G : V \times I \to V$ a C^∞ mapping and $g : I \to V$ a smooth mapping, where I is an open interval in R containing zero. Consider the equation*

$$s\frac{df}{ds} + Nf = sG(s, f(s)) = g(s) \tag{4.11}$$

for a function f defined on a neighbourhood of 0 and I and taking values in V. Suppose that each eigenvalue of N has a positive real part. Then there exists an open Interval J with $0 \in J \subset I$ and a unique bounded C^1 function f on $J \setminus 0$ satisfying (4.11). Moreover f extends to a C^∞ solution of (4.11) on J if N, G and g depend smoothly on a parameter z and the eigenvalues of N are distinct then the solution also depends smoothly on z.

For a given equation of state equations (2.16) and (2.18) are forming a system of differential equations in $P(r)$ and $w(r)$. The aim is to apply theorem 1.

Recall (2.18) and (2.16) with $x = r^2$ and $dr = \frac{dx}{2\sqrt{x}}$. These substitutions give

$$x\frac{dw}{dx} + \frac{3}{2}w = 2\pi\rho \tag{4.12}$$

$$\frac{d\rho}{dx} = -\left(\frac{dP}{d\rho}\right)^{-1}\frac{1}{2}\frac{\left(4\pi P + w - \frac{\Lambda}{3}\right)(P+\rho)}{1 - 2wx - \frac{\Lambda}{3}x}. \tag{4.13}$$

Looking for the mathematical structure of the equations one has that the first one is singular at the centre $x = 0$ whereas the second one is regular there. Both equations have different properties, but using the substitution

$$\rho = \rho_c + x\rho_1,$$

where ρ_c is the value of the central density, the second equation becomes singular at the centre, too. The transformed system is

$$x\frac{dw}{dx} + \frac{3}{2}w = 2\pi\rho_c + 2\pi x\rho_1 \tag{4.14}$$

$$x\frac{d\rho_1}{dx} + \rho_1 = -\left(\frac{dP}{d\rho}\right)^{-1}\frac{1}{2}\frac{\left(4\pi P + w - \frac{\Lambda}{3}\right)(P+\rho_c + x\rho_1)}{1 - 2wx - \frac{\Lambda}{3}x}. \tag{4.15}$$

The system will be put in the required form to apply theorem 1. After substituting in $\rho = \rho_c + x\rho_1$ and the corresponding pressure relation $P = P_c + xP_1(\rho_1)$, only one more algebraic relation is needed. It is easily verified that

$$\left(1 - 2wx - \frac{\Lambda}{3}x\right)^{-1} = 1 + \left(2wx + \frac{\Lambda}{3}x\right)\left(1 - 2wx - \frac{\Lambda}{3}x\right)^{-1}.$$

Then the matrix N has the form

$$\begin{pmatrix} 3/2 & 0 \\ \frac{(P_c+\rho_c)}{2dP/d\rho(\rho_c)} & 1 \end{pmatrix}. \tag{4.16}$$

It is easy to obtain that N does not change if it is written in terms of effective values.

Now theorem 1 can be applied. Thus the system has a unique bounded solution in the neighbourhood of the centre, which is C^∞. Therefore one has a unique smooth solution to (2.16) and (2.18) near the centre.

This disproves Collins [3] statement that for given equation of state, central pressure and cosmological constant the solution is not uniquely determined.

Standard theorems for differential equations imply that the solution can be extended as long as the right-hand sides of (2.16) or (4.13) are well defined. If the pressure satisfies $P < \infty$ and the denominator of (4.13) does not vanish, this means $y = 1 - 2wx - (\Lambda/3)x > 0$, then the right-hand sides are well defined. The second term involve the cosmological constant and therefore some new features will arise.

Uniqueness of the solution at the centre implies the following theorem.

Theorem 2 *Suppose $\rho(P)$, P_c and the cosmological constant Λ are given such that*

$$4\pi P_c + \frac{4\pi}{3}\rho(P_c) - \frac{\Lambda}{3} = 0, \tag{4.17}$$

note that $(4\pi/3)\rho(P_c) = w_c$. Then the solution is the Einstein static universe with $\Lambda_E = \Lambda$.

4.3 Extension of the solution

Theorem 3 *Assume the pressure is decreasing near the centre, this means*

$$4\pi P_c + \frac{4\pi}{3}\rho(P_c) - \frac{\Lambda}{3} > 0. \tag{4.18}$$

Then the solution is extendible and the pressure is monotonically decreasing if

$$4\pi P + w - \Lambda/3 > 0. \tag{4.19}$$

Proof. Suppose that $\rho = \rho(P)$, P_c and Λ are given such that P is decreasing near the centre, then $w(x)_{,x} \leq 0$. Using $y > 0$ because of the metric's signature, (4.10) implies

$$\left(y\zeta_{,x}\right)_{,x} \leq 0. \tag{4.20}$$

Rewriting (4.4) gives

$$y\zeta_{,x} = \frac{\zeta}{2y}\left(4\pi P + w - \frac{\Lambda}{3}\right), \tag{4.21}$$

next using the implication of (4.20) leads to

$$y\zeta_{,x} \le (y\zeta_{,x})(0).$$

Together with the explicit expression of $y\zeta_{,x}$ in (4.21) this finally shows

$$y \ge \frac{4\pi P + w - \frac{\Lambda}{3}}{4\pi P_c + w_c - \frac{\Lambda}{3}}. \tag{4.22}$$

Therefore the new Buchdahl variable y cannot vanish before the numerator does. Thus the right-hand sides of (2.16) or (4.13) are well defined and one can extend the solution if $4\pi P + w - \Lambda/3 > 0$.

Since $y \ge 0$ and $4\pi P + w - \Lambda/3 > 0$ the sign of the right-hand side of (4.13) is strictly negative. Therefore the density and by the equation of state the pressure are monotonically decreasing functions.

4.4 A possible coordinate singularity $y = 0$

For some cosmological constants the new Buchdahl variable y may vanish before the pressure does. In the constant density case the vanishing of y corresponded to a coordinate singularity and one could extend the solution. Therefore one would like to show that this is also true for a prescribed equation of state.

The following theorem only indicates that the vanishing of y corresponds to a coordinate singularity. The square of the Riemann tensor in Buchdahl variables (B.6) is given in appendix B.

Theorem 4 *The squared Riemann tensor (B.6) does not diverge as $y \to 0$ if the pressure is finite.*

Proof. This is seen by first showing that $y \to 0$ implies $4\pi P + w - \Lambda/3 \to 0$, which is the numerator of the TOV-Λ equation (2.16) and of (4.22).

Assume that $\rho(P)$, P_c and Λ are given such that the pressure is decreasing near the centre. Assume further that Λ is given such that y vanishes before the pressure. Then there exists $r_s > 0$ such that $4\pi P_s + w_s - \Lambda/3 = 0$. Show that $r_s = \hat{r}$, where $\hat{r}$ denotes the radius where the TOV-Λ equation becomes singular. This is proved indirectly:

Assume $r_s \neq \hat{r}$, i.e. $y_s \neq 0$. Then $4\pi P_s + w_s - \Lambda/3 = 0$ implies $P_s' = 0$ and equation (4.21) gives

$$\zeta_{,x}(x_s) = 0. \tag{4.23}$$

Evaluating (4.9) at x_s leads to

$$y_s^2\zeta_{,xx}(x_s) - \frac{1}{2}\zeta(x_s)w_{,x}(x_s) = 0. \tag{4.24}$$

The pressure is assumed to be decreasing near the centre, thus $w_{,x}(x_s) \leq 0$. Since $y_s \neq 0$ (4.24) implies

$$\zeta_{,xx}(x_s) \leq 0. \tag{4.25}$$

Using (4.2), the definition of ζ, implies that $P''(r_s) \geq 0$. On the other hand the above assumptions on the pressure are such that $P''(r)$ is smaller than zero near the centre. Therefore $P''(r) \leq 0$ on $0 < r \leq r_s$. And both inequalities together imply $P''(r_s) = 0$, which using (4.2) combines to

$$\zeta_{,xx}(x_s) = 0. \tag{4.26}$$

Evaluating (4.9) at x_s using (4.23) and (4.26) finally gives

$$\zeta(x_s)w_{,x}(x_s) = 0. \tag{4.27}$$

Since ζ cannot vanish $w_{,x}(x_s) = 0$. It was assumed that $r_s, x_s > 0$ thus equation (4.12) reads

$$0 = x_s w_{,x}(x_s) = \frac{3}{2}\left(\frac{4\pi}{3}\rho_s - w_s\right), \tag{4.28}$$

which gives $w_s = (4\pi/3)\rho_s$. Therefore $\rho(r) = \rho_s = \text{const.}$ on $0 < r \leq r_s$. Conversely $P(r) = \text{const.}$, which contradicts the above assumption that the pressure is decreasing near the centre. Thus $P'(r_s) \neq 0$, therefore y has to vanish and $r_s = \hat{r}$. So the situation is similar to the constant density case. Therefore the expression

$$y\frac{\zeta_{,x}}{\zeta} = -\frac{1}{2}\left(\frac{4\pi P + w - \frac{\Lambda}{3}}{y}\right) \tag{4.29}$$

does not diverge as $y \to 0$ and shows finiteness of the squared Riemann tensor.

The first term of (B.6) gives $4/x^2$, the second and the third term vanish because (4.29) implies $y^2\zeta_{,x}/\zeta = 0$. By the above the last term is bounded if the pressure is finite.

This indicates that $y = 0$ corresponds to a coordinate singularity.

At the possible coordinate singularity the pressure has a well defined value given by

$$P(\hat{r}) = \frac{1}{4\pi}\left(\frac{\Lambda}{3} - w(\hat{r})\right). \tag{4.30}$$

4.5 Existence of global solutions with $\Lambda < 4\pi\rho_b$

The following theorem shows the existence of global solutions. Two possibilities occur. Either the matter occupies the whole space or the pressure vanishes at a finite radius, in which case a vacuum solution is joined.

Theorem 5 *Let an equation of state be given such that ρ is defined for $p \geq 0$, non-negative and continuous for $p \geq 0$, C^∞ for $p > 0$ and suppose that $d\rho/dp > 0$ for $p > 0$. Further let the cosmological constant be given such that $\Lambda < 4\pi\rho_b$.*

Then the pressure is decreasing near the centre and there exists for any positive value of central pressure P_c a unique inextendible static, spherically symmetric solution of Einstein's field equations with cosmological constant with a perfect fluid source and equation of state $\rho(P)$.

If $\Lambda \leq 0$ then the matter either occupies the whole space with ρ tending to ρ_∞ as r tends to infinity or the matter has finite extend. In the second case a unique Schwarzschild-anti-de Sitter solution is joined on as an exterior field.

If $0 < \Lambda < 4\pi\rho_b$ the matter always has finite extend and a unique Schwarzschild-de Sitter solution is joined on as an exterior field.

Proof. If the cosmological constant is given such that $\Lambda < 4\pi\rho_b$ then

$$\begin{aligned} 0 &< \frac{4\pi}{3}\rho_b - \frac{\Lambda}{3} \\ &< 4\pi P_c + \frac{4\pi}{3}\rho(P_c) - \frac{\Lambda}{3}, \end{aligned}$$

and the pressure is decreasing near the centre, see (4.18).

Since the pressure is decreasing near the centre the denominator of (4.22) is a positive number $\mathcal{D}$. Then the numerator of (4.22) can be estimated by

$$\begin{aligned} y &\geq \frac{4\pi P + w - \frac{\Lambda}{3}}{\mathcal{D}} \\ &\geq \frac{w_b - \frac{\Lambda}{3}}{\mathcal{D}} \qquad (4.31) \\ &\geq \frac{\frac{4\pi}{3}\rho_b - \frac{\Lambda}{3}}{\mathcal{D}}. \qquad (4.32) \end{aligned}$$

Thus if

$$\Lambda < 4\pi\rho_b, \qquad (4.33)$$

then[2] the new Buchdahl variable cannot vanish before the pressure, ρ_b is given by the equation of state because $\rho_b = \rho(P = 0)$. The coordinate x_b where the pressure vanishes will be taken as the definition of the stellar object's radius r_b.

$\Lambda \leq 0$

If $\Lambda \leq 0$ the matter can occupy the whole space because (4.22) implies positivity of y.

Suppose $P(x_b) = 0$. At the corresponding radius r_b the Schwarzschild-anti-de Sitter solution, (2.22) with negative cosmological constant, is joined uniquely by the condition $M = m(r_b)$. In this manner the metric is C^0 only, because the density at the boundary may be non-zero. The metric is C^1 at $P(r_b) = 0$ if Gauss coordinates relative to the hypersurface $P(r_b) = 0$ are used. These are given by $\chi(r) = \int_0^r e^{a(s)/2} ds$. If the boundary density does not vanish the Ricci tensor has a discontinuity. Thus the metric is at most C^1. Without further assumptions on the boundary density this cannot be improved.

Assume now that $P(x) > 0$ for all $x > 0$. $P(x)$ is monotonically decreasing, therefore $\lim_{x\to\infty} P(x) = P_\infty$ exists. This implies that the pressure gradient tends to zero as $x \to \infty$. Because of $y^{-1} \to 0$ as $x \to \infty$ equation (2.16) does not imply that $P_\infty = 0$, which it does if $\Lambda = 0$. Thus the equation of state only gives $\rho \to \rho_\infty = \rho(P_\infty)$ as $x \to \infty$.

$0 < \Lambda < 4\pi\rho_b$

If $0 < \Lambda < 4\pi\rho_b$ then one can estimate the pressure (4.30) at the possible coordinate singularity $y = 0$. Note that

$$\frac{\Lambda}{3} < \frac{4\pi}{3}\rho_b \leq w_b \leq w(r), \tag{4.34}$$

which holds for all r. Therefore

$$P(\hat{r}) = \frac{1}{4\pi}\left(\frac{\Lambda}{3} - w(\hat{r})\right) < 0. \tag{4.35}$$

Hence there exists r_b such that $P(r_b) = 0$. Since the pressure is decreasing and $P(\hat{r}) < 0$ it follows that $r_b < \hat{r}$.

Thus if the cosmological constant is positive and $\Lambda < 4\pi\rho_b$ then the pressure always vanishes at some x_b. At the corresponding r_b the Schwarzschild-de Sitter solution is joined uniquely by the same condition $M = m(r_b)$. The

[2] The argument of [14] is applicable if the cosmological constant satisfies this condition.

metric is at most C^1 because the boundary density is larger than zero, this cannot be improved.

4.6 Generalised Buchdahl inequality

For stellar models an analogous Buchdahl inequality is derived. It turns out that the new inequality nearly coincides with the former (3.34), which was derived for constant density.

The proof of the following theorem solves the field equations with constant density written in Buchdahl variables. This solution is compared with a decreasing solution and leads to an analogous Buchdahl inequality.

The solution with constant density is denoted with a tilde above.

Theorem 6 *Let the cosmological constant be given such that $\Lambda < 4\pi\rho_b$. Then for stellar models there holds*

$$\sqrt{1 - 2w_b r_b^2 - \frac{\Lambda}{3} r_b^2} \geq \frac{1}{3} - \frac{\Lambda}{9w_b}. \tag{4.36}$$

Proof. Assume that $\rho(P)$, P_c and Λ are given such that the pressure is decreasing near the centre. In section 4.3 it was shown that this means that pressure and mean density are decreasing functions.

Equation (4.10) implies

$$\begin{aligned} (\tilde{y}\tilde{\zeta}_{,x})_{,x} &= 0 \\ \tilde{y}\tilde{\zeta}_{,x} &= D, \end{aligned} \tag{4.37}$$

where D is a constant of integration.

If the density is constant ζ is the only unknown function in Einstein's field equations.

The function ζ can be normalised to give $\zeta_b = y_b$. The constant D is obtained from (4.4) evaluated at the boundary, therefore

$$D = \frac{1}{2}\tilde{w} - \frac{\Lambda}{6}, \tag{4.38}$$

which can be used to integrate (4.37). Notice that (4.7) reads

$$-2\tilde{y}\tilde{y}_{,x} = 2\tilde{w} + \frac{\Lambda}{3}.$$

The right-hand side of (4.37) can be written as

$$\tilde{y}\tilde{\zeta}_{,x} = \left(2\tilde{w} + \frac{\Lambda}{3}\right)\left(\frac{1}{2}\tilde{w} - \frac{\Lambda}{6}\right)\left(2\tilde{w} + \frac{\Lambda}{3}\right)^{-1}.$$

Substitute in $-2\tilde{y}\tilde{y}_{,x}$ because of the first factor and divide by $\tilde{y}$. Integration then leads to

$$\tilde{\zeta}(x) = -\frac{\tilde{w} - \frac{\Lambda}{3}}{2\tilde{w} + \frac{\Lambda}{3}}\tilde{y}(x) + \text{const.}$$

The constant of integration is

$$\text{const.} = \tilde{\zeta}(0) + \frac{\tilde{w} - \frac{\Lambda}{3}}{2\tilde{w} + \frac{\Lambda}{3}}.$$

So the solution of differential equation (4.37) is obtained. Equation (4.20) implies

$$\tilde{y}\tilde{\zeta}_{,x} = (y\zeta_{,x})_b < y\zeta_{,x}.$$

Using that $\tilde{y} > y$ one finds

$$\zeta(x) \geq \tilde{\zeta}(x) = -\frac{\tilde{w} - \frac{\Lambda}{3}}{2\tilde{w} + \frac{\Lambda}{3}}\tilde{y}(x) + \tilde{\zeta}(0) + \frac{\tilde{w} - \frac{\Lambda}{3}}{2\tilde{w} + \frac{\Lambda}{3}}.$$

Evaluating this at the boundary and using the normalisation condition it is found that

$$y_b \geq -\frac{\tilde{w} - \frac{\Lambda}{3}}{2\tilde{w} + \frac{\Lambda}{3}}y_b + \tilde{\zeta}(0) + \frac{\tilde{w} - \frac{\Lambda}{3}}{2\tilde{w} + \frac{\Lambda}{3}}.$$

Since $\tilde{\zeta}(0)$ is positive some algebra leads to the analogous Buchdahl inequality

$$y_b \geq \frac{1}{3} - \frac{\Lambda}{9\tilde{w}}. \tag{4.39}$$

One compares solutions with decreasing mean density and solution with constant density. The compared constant density corresponds to the boundary mean density. Therefore (4.39) reads

$$y_b \geq \frac{1}{3} - \frac{\Lambda}{9w_b}, \tag{4.40}$$

which nearly equals the analogous Buchdahl inequality (3.34) if the expression for y_b is used. The only difference is that the equation contains the

boundary mean density w_b rather than the boundary density ρ_b. Explicitly written out leads to

$$\sqrt{1 - 2w_b r_b^2 - \frac{\Lambda}{3} r_b^2} \geq \frac{1}{3} - \frac{\Lambda}{9w_b}, \tag{4.41}$$

and holds for all monotonically decreasing densities and proves theorem 6.

Equation (4.34) implies that the right-hand sides of the last two equations (4.41) and (4.40) are positive. This improves (4.22), which only gave $y > 0$. Solving inequality (4.41) for r_b gives an analogue of (3.35)

$$r_b^2 \leq \frac{\frac{1}{9}\left(4 - \frac{\Lambda}{3w_b}\right)}{w_b}. \tag{4.42}$$

As before the only difference is that the boundary mean density is used instead of the boundary density itself. Multiply (4.42) by w_b and use that $w_b = m(r_b)/r_b^3$ gives

$$\frac{m(r_b)}{r_b} \leq \frac{1}{9}\left(4 - \frac{\Lambda r_b^3}{3m(r_b)}\right).$$

Taking all terms on one side and multiplying by $9m(r_b)r_b$ leads to

$$9m(r_b)^2 - 4m(r_b)r_b + \frac{\Lambda}{3} r_b^4 \leq 0.$$

Rewriting this with

$$9m(r_b)^2 - 4m(r_b)r_b = \left(3m(r_b) - \frac{2}{3}r_b\right)^2 - \frac{4}{9}r_b^2,$$

finally gives

$$\left(3m(r_b) - \frac{2}{3}r_b\right)^2 - r_b^2\left(\frac{4}{9} - \frac{\Lambda}{3}r_b^2\right) \leq 0. \tag{4.43}$$

Another way of writing the above (4.43) is

$$\left(3m(r_b) - \frac{2}{3}r_b - r_b\sqrt{\frac{4}{9} - \frac{\Lambda}{3}r_b^2}\right)\left(3m(r_b) - \frac{2}{3}r_b + r_b\sqrt{\frac{4}{9} - \frac{\Lambda}{3}r_b^2}\right) \leq 0, \tag{4.44}$$

where $a^2 - b^2 = (a-b)(a+b)$ was used. Putting $\Lambda = 0$ shows that the second term has to be positive because of positivity of the mass. Then the first term is negative and implies the Buchdahl inequality (3.38). For negative cosmological constants the square root terms are always well defined. For positive cosmological constants they are well defined if

$$r_b \leq \sqrt{\frac{4}{3}} \frac{1}{\sqrt{\Lambda}}. \tag{4.45}$$

Then both expressions of the product (4.44) can be combined to

$$3m(r_b) \leq \frac{2}{3} r_b + r_b \sqrt{\frac{4}{9} - \frac{\Lambda}{3} r_b^2}. \tag{4.46}$$

With $\Lambda = 0$ equation (4.46) directly implies Buchdahl's inequality (3.39). It holds for arbitrary static fluid balls in which the density does not increase outwards.

4.7 Solutions without singularities

The last two sections showed the existence of stellar models for cosmological constants $\Lambda < 4\pi\rho_b$. Equation (4.22) implies that the boundary of the stellar object has a lower bound given by the black-hole event horizon and an upper bound given by the cosmological event horizon. The upper bound only occurs if the cosmological constant is positive, as already said.

Stellar models with $\Lambda \leq 0$ have a lower bound given by the black-hole event horizon. At the boundary the Schwarzschild-anti-de Sitter solution is joined on as an exterior field. Figure 29 shows that this joined exterior field has no singularities. Therefore stellar models with $\Lambda \leq 0$ have no singularities. Solutions with cosmological constant satisfying $\Lambda \leq 0$ are globally static.

For positive cosmological constants the situation is different. But one can construct solutions without singularities.

At the boundary $r = r_b$ the Schwarzschild-de Sitter solution is joined C^1 by the usual procedure introducing Gauss coordinates. This leads to Penrose-Carter figure 30. This spacetime has an infinite sequence of singularities $r = 0$ and space-like infinities $r = \infty$.

The surface $r = r_b$ can also be found in the vacuum region where the time-like Killing vector is past directed. This means that a second stellar object is put in the spacetime. Therefore Penrose-Carter figure 31 gives a general picture and is not restricted to constant density solutions.

Note that solutions with positive cosmological constant are not globally static because of the dynamical parts of the exterior Schwarzschild-de Sitter solution.

Solutions without singularities and without horizons were recently described by [15]. They considered an interior de Sitter region and an exterior Schwarzschild solution separated by a small shell of matter with equation of state $\rho(P) = P$. The shell replaces both the de Sitter and the Schwarzschild horizon. The new solution has no singularities.

4.8 Remarks on finiteness of the radius

So far it has been shown that given an equation of state, a central pressure and a cosmological constant there exist a unique model with finite or infinite extend. This depends on the given equation of state and on the cosmological constant. This section gives criteria to distinguish these two cases, see [19].

$\Lambda > 0$

If $\Lambda > 0$ solutions are always finite. Thus given an equation of state $\rho = \rho(P)$ and cosmological constant such that

$$\Lambda < 4\pi\rho(P = 0) = 4\pi\rho_b, \tag{4.47}$$

then there always exist a radius r_b where the pressure vanishes.

$\Lambda \leq 0$

If $\Lambda \leq 0$ either the pressure vanishes for some finite radius or the density is always positive and tends to ρ_∞ as r tends to infinity.

First a necessary condition for finiteness of the radius is derived. Recall (2.16) written in the variable x

$$P_{,x} = -\frac{1}{2}\frac{\left(4\pi P + w - \frac{\Lambda}{3}\right)(P+\rho)}{1 - 2wx - \frac{\Lambda}{3}x}. \tag{4.48}$$

Assume that the stellar model is bounded by x_b or the corresponding r_b. On the closed interval $[0, x_b]$ the function y has a positive minimum K. The maximal contribution due to the first factor is given by its value at the centre. Therefore

$$P_{,x} > -L\,(P+\rho)\,,$$

where the constant L is given by

$$L = \frac{1}{2K}\left(4\pi P_c + w_c - \frac{\Lambda}{3}\right). \tag{4.49}$$

Compare the above inequality with

$$P^*_{,x} = L\left(P^* + \rho^*\right),$$

which can be integrated to give

$$\int_0^{P_c^*} \frac{dP}{P+\rho} = Lx_b = Lr_b^2.$$

Thus we obtain a bound for the integral. Hence if the star has finite boundary then

$$\int_0^{P_c} \frac{dP}{P+\rho} < \infty. \tag{4.50}$$

This gives a necessary condition for the equation of state to give fluid balls of finite extend. Then equation (2.17) implies that $\nu(r)$ is finite.

Now a sufficient condition is derived. Since the density distribution is decreasing $m \geq \frac{4\pi}{3}\rho r^3$, $w \geq \frac{4\pi}{3}\rho$ for all $r > 0$.

Thus (4.48) gives

$$P_{,x} < -\frac{1}{2}w\rho < -\frac{2\pi}{3}\rho^2, \tag{4.51}$$

where one needed that

$$4\pi P - \frac{\Lambda}{3} > 0,$$

which is always achieved as long as $\Lambda \leq 0$. Then comparing with a P^* again and using that there has to exist a point x_b where the pressure vanishes one may integrate (4.51). Therefore if

$$\int_0^{P_c} \frac{dP}{\rho^2} < \infty, \tag{4.52}$$

the stellar object has finite radius. Both criteria (4.50), (4.52) only involve the low pressure behaviour of the equation of state.

Appendices

A Einstein tensor and energy-momentum conservation

In the following a static, spherically symmetric metric is considered. In the used notation it may be written as

$$ds^2 = -\frac{1}{\lambda}e^{\lambda\nu(r)}dt^2 + e^{a(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta \; d\phi^2). \tag{A.1}$$

The non vanishing Christoffel symbols are:

$$\begin{aligned}
\Gamma^t_{tr} &= \frac{1}{2}\lambda\nu'(r) & & \\
\Gamma^r_{tt} &= \frac{1}{2}\nu'(r)e^{\lambda\nu(r)-a(r)} & \Gamma^r_{rr} &= \frac{1}{2}a'(r) \\
\Gamma^r_{\theta\theta} &= -re^{-a(r)} & \Gamma^r_{\phi\phi} &= -r\sin^2\theta e^{-a(r)} \\
\Gamma^\theta_{\theta r} &= \frac{1}{r} & \Gamma^\theta_{\phi\phi} &= -\cos\theta\sin\theta \\
\Gamma^\phi_{\phi r} &= \frac{1}{r} & \Gamma^\phi_{\phi\theta} &= \cot\theta.
\end{aligned}$$

The trace terms are:

$$\begin{aligned}
\Gamma^\sigma_{t\sigma} &= 0 & \Gamma^\sigma_{r\sigma} &= \frac{2}{r} + \frac{1}{2}(\lambda\nu'(r) + a'(r)) \\
\Gamma^\sigma_{\theta\sigma} &= \cot\theta & \Gamma^\sigma_{\phi\sigma} &= 0.
\end{aligned}$$

The Ricci tensor is defined by

$$R_{\mu\nu} = \Gamma^\rho_{\mu\nu,\rho} - \Gamma^\rho_{\mu\rho,\nu} + \Gamma^\sigma_{\mu\nu}\Gamma^\rho_{\sigma\rho} - \Gamma^\sigma_{\mu\rho}\Gamma^\rho_{\sigma\nu}. \tag{A.2}$$

Its components are:

$$\begin{aligned}
R_{tt} &= e^{\lambda\nu(r)-a(r)}\left(\frac{1}{2}\nu''(r) + \frac{1}{4}\lambda\nu'(r)^2 + \frac{1}{r}\nu'(r) - \frac{1}{4}a'(r)\nu'(r)\right) \\
R_{rr} &= -\frac{1}{2}\lambda\nu''(r) - \frac{1}{4}\lambda^2\nu'(r)^2 + \frac{1}{4}a'(r)\lambda\nu'(r) + \frac{1}{r}a'(r) \\
R_{\theta\theta} &= 1 - e^{-a(r)} + \frac{1}{2}ra'(r)e^{-a(r)} - \frac{1}{2}r\lambda\nu'(r)e^{-a(r)} \\
R_{\phi\phi} &= \sin^2\theta R_{\theta\theta} \\
R_{\mu\nu} &= 0 \;\text{ if } \mu \neq \nu.
\end{aligned}$$

The Ricci scalar is

$$R = g^{\mu\nu} R_{\mu\nu} = -\lambda\nu''(r)e^{-a(r)} - \frac{1}{2}\lambda^2\nu'(r)^2 e^{-a(r)} + \frac{1}{2}a'(r)\lambda\nu'(r)e^{-a(r)} + \frac{2}{r^2} - \frac{2e^{-a(r)}}{r^2} + \frac{2}{r}a'(r)e^{-a(r)} - \frac{2}{r}\lambda\nu'(r)e^{-a(r)}.$$

The Einstein tensor is defined by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}. \tag{A.3}$$

Only its four diagonal components remain. They are given by:

$$G_{tt} = \frac{1}{\lambda r^2}e^{\lambda\nu(r)}\frac{d}{dr}\left(r - re^{-a(r)}\right) \tag{A.4}$$

$$G_{rr} = \frac{1}{r^2}\left(1 + r\lambda\nu'(r) - e^{a(r)}\right) \tag{A.5}$$

$$G_{\theta\theta} = r^2 e^{-a(r)}\frac{1}{2}(\lambda\nu''(r) - \frac{1}{r}a'(r) + \frac{1}{r}\lambda\nu'(r) + \frac{1}{2}\lambda^2\nu'(r)^2 - \frac{1}{2}a'(r)\lambda\nu'(r)) \tag{A.6}$$

$$G_{\phi\phi} = \sin^2\theta G_{\theta\theta}. \tag{A.7}$$

The Einstein tensor has vanishing covariant derivative and therefore implies energy-momentum conservation,

$$\nabla_\nu T^{\mu\nu} = 0. \tag{A.8}$$

For a perfect fluid in a static, spherically symmetric spacetime the energy-momentum tensor has the form

$$T_{\mu\nu} = (\rho(r) + \lambda P(r))U_\mu U_\nu + P(r)g_{\mu\nu}, \tag{A.9}$$

where 4-velocity $U_t = (-1/\lambda)e^{\lambda\nu(r)/2}$ is given for a fluid at rest. Conservation of this quantity gives four equations of motion. Because of symmetries, only the radial component $\mu = r$ does not equal zero. By using the Christoffel symbols derived above, it is found that

$$0 = (\rho(r) + \lambda P(r))\,\frac{\nu'(r)}{2}e^{-a(r)} + P'(r)e^{-a(r)},$$

which gives

$$2P'(r) = -(\lambda P(r) + \rho(r))\nu'(r). \tag{A.10}$$

B Geometric invariants

Constant density solutions

Metric (3.2) of the constant density solutions for $\Lambda > -8\pi\rho_0$ can be written

$$ds^2 = -\left(\frac{P_c + \rho_0}{P(\alpha) + \rho_0}\right)^2 dt^2 + \frac{1}{k}(d\alpha^2 + \sin^2\!\alpha \; d\Omega^2). \tag{B.1}$$

Integration of (3.1) gives

$$P(\alpha) = \rho_0 \frac{\left(\frac{\Lambda}{4\pi\rho_0} - 1\right) + C\cos\alpha}{3 - C\cos\alpha}. \tag{B.2}$$

From this one can find the square of the Riemann tensor. It reads

$$\begin{aligned} R_{\mu\nu\lambda\kappa}R^{\mu\nu\lambda\kappa} &= 12\left(\frac{8\pi}{3}\rho_0 + \frac{\Lambda}{3}\right)^2 \\ &\quad \times \frac{81 - C^2\cos^2\alpha\left(9 - 6C\cos\alpha - 2C^2\cos^2\alpha\right)}{\left(3 + C\cos\alpha\right)^2\left(3 - C\cos\alpha\right)^2}, \end{aligned} \tag{B.3}$$

and gives two important implications.

First it shows that the radial coordinate r behaves badly as $r \to \hat{r}$, which corresponds to $\alpha \to \pi/2$. Thus

$$R_{\mu\nu\lambda\kappa}R^{\mu\nu\lambda\kappa}(\alpha = \pi/2) = 12\left(\frac{8\pi}{3}\rho_0 + \frac{\Lambda}{3}\right)^2, \tag{B.4}$$

and one clearly has a coordinate singularity because the square of Riemann tensor is finite. It was already pointed out that this is only a coordinate singularity when the geometry of the interior solutions was discussed.

Secondly, the square of the Riemann tensor diverges as the pressure diverges. Compare the denominators of (B.2) and (B.3). Thus there is a geometric singularity if the pressure is divergent.

Buchdahl variables

In Buchdahl variables

$$\begin{aligned} y^2 &= e^{-a(r)} = 1 - 2w(r)r^2 - \frac{\Lambda}{3}r^2, \\ \zeta &= e^{\nu/2}, \; x = r^2, \end{aligned}$$

the static spherically symmetric metric can be written

$$ds^2 = -\zeta(x)^2 dt^2 + \frac{dx^2}{4xy(x)^2} + x(d\theta^2 + \sin^2 d\phi^2). \tag{B.5}$$

In these coordinates the square of the Riemann tensor becomes

$$\begin{aligned} R_{\mu\nu\lambda\kappa}R^{\mu\nu\lambda\kappa} = & \left[2\,\frac{1-y^2}{x}\right]^2 + 2\left[2\left(y^2\right)_{,x}\right]^2 \\ & + 2\left[y^2\frac{\zeta_{,x}}{\zeta}\right]^2 + \left[\frac{\left(\zeta_{,x}^2\, xy^2\right)_{,x}}{\zeta\zeta_{,x}}\right]^2. \end{aligned} \tag{B.6}$$

The square of the Weyl tensor is given by

$$C_{\mu\nu\lambda\kappa}C^{\mu\nu\lambda\kappa} = \frac{64}{3}\frac{y^2}{\zeta^2}\left(\left(y\zeta_{,x}\right)_{,x} + \frac{1}{2}\frac{w_{,x}\zeta}{y}\right). \tag{B.7}$$

Conformal flatness of constant density solutions

In chapter 4 Einstein's field equation with cosmological constant for a perfect fluid were rewritten in terms of Buchdahl variables. This gave equation (4.10), stated again

$$\left(y\zeta_{,x}\right)_{,x} - \frac{1}{2}\frac{w_{,x}\zeta}{y} = 0. \tag{B.8}$$

If the density is assumed to be constant then $w_{,x} = 0$. Therefore equation (B.8) implies that the square of the Weyl tensor (B.7) vanishes. In spherical symmetry this then implies that the Weyl tensor vanishes. Thus constant density solutions are conformally flat. Furthermore it follows that this is not true if $w_{,x} \neq 0$.

Invariants for other metrics

To complete this appendix the invariants of the Schwarzschild-de Sitter (2.22) and Nariai (3.48) solution are given. For Schwarzschild-de Sitter they are given by

$$R_{\mu\nu\lambda\kappa}R^{\mu\nu\lambda\kappa} = 48\frac{M^2}{r^6} + \frac{8}{3}\Lambda^2 \tag{B.9}$$

$$C_{\mu\nu\lambda\kappa}C^{\mu\nu\lambda\kappa} = 48\frac{M^2}{r^6}. \tag{B.10}$$

The Nariai metric gives

$$R_{\mu\nu\lambda\kappa}R^{\mu\nu\lambda\kappa} = 8\Lambda^2 \tag{B.11}$$

$$C_{\mu\nu\lambda\kappa}C^{\mu\nu\lambda\kappa} = \frac{16}{3}\Lambda^2. \tag{B.12}$$

C Penrose-Carter diagrams

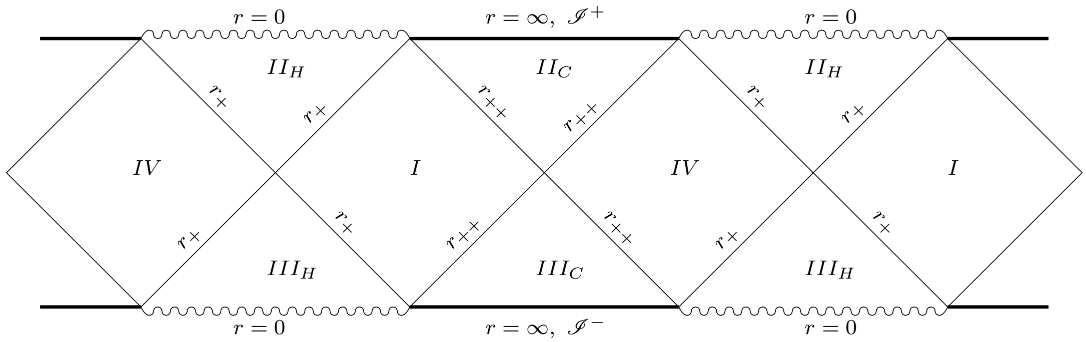

Figure 28: Penrose-Carter diagram for Schwarzschild-de Sitter space. The Killing vector $\partial/\partial t$ is time-like and future-directed in regions I and time-like and past-directed in regions IV. In the others regions it is space-like. The surfaces $r = r_+$ and $r = r_{++}$ are black-hole and cosmological event horizons, respectively. $\mathscr{I}^+$ and $\mathscr{I}^-$ are the space-like infinities.

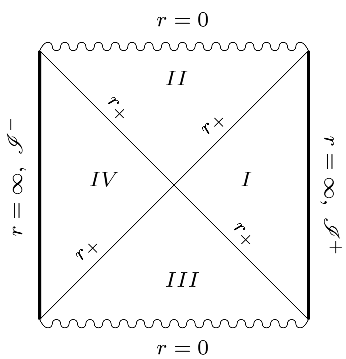

Figure 29: Penrose-Carter diagram for Schwarzschild-anti-de Sitter space. The surface $r = r_+$ is the black-hole event horizons. $\mathscr{I}^+$ and $\mathscr{I}^-$ are the time-like infinities.

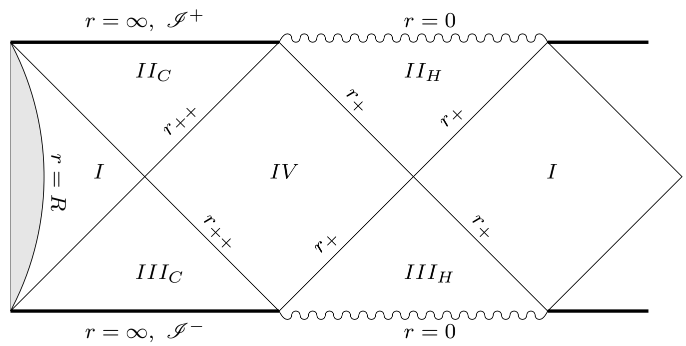

Figure 30: Penrose-Carter diagram with one stellar object having a radius R which lies between the two horizons. The group orbits are increasing at the boundary. The shaded region is the matter solution with regular centre. There is still an infinite sequence of singularities $r = 0$ and space-like infinities $r = \infty$.

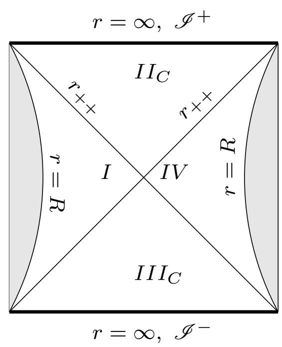

Figure 31: Penrose-Carter diagram with two stellar objects having radii R which lie between the two horizons. Since the group orbits are increasing up to R the vacuum part contains the cosmological event horizon r_{++}. This solution with two objects has no singularities. Because of regions II_C and III_C this spacetime is not globally static.

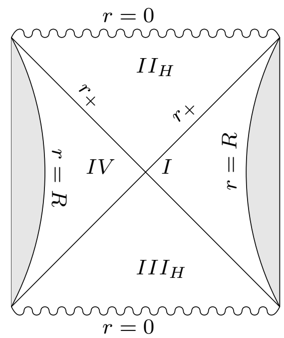

Figure 32: Penrose-Carter diagram with two stellar objects having radii R which lie between the two horizons. The group orbits of the interior solutions are decreasing where the vacuum solution is joined. Thus the $r = 0$ singularity of the vacuum part is present.

References

[1] T. W. Baumgarte and A. D. Rendall. Regularity of spherically symmetric static solutions of the Einstein equations. *Class. Quant. Grav.*, 10:327–332, 1993.

[2] H. A. Buchdahl. General relativistic fluid spheres. *Phys. Rev.*, 116:1027–1034, 1959.

[3] C. B. Collins. Comments on the static spherically symmetric cosmologies of Ellis, Maartens, and Nel. *J. Math. Phys.*, 24:215–219, 1983.

[4] J. Ehlers. The Newtonian limit of general relativity. preprint 89/1, Dept. of Appl. Math., University of Cape Town.

[5] J. Ehlers. *The Newtonian limit of general relativity.* In Ferrarese [8], 1991.

[6] J. Ehlers. Examples of Newtonian limits of relativistic spacetimes. *Class. Quant. Grav.*, 14:A119–A126, 1997.

[7] Albert Einstein. Cosmological considerations in the general theory of relativity. *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)*, 1917:142–152, 1917.

[8] G. Ferrarese, editor. *Classical Mechanics and Relativity: Relationship and Consistency.* American Institute of Physics, 1991.

[9] G. W. Gibbons and S. W. Hawking. Cosmological event horizons, thermodynamics, and particle creation. *Phys. Rev.*, D15:2738–2751, 1977.

[10] Geyer K. H. Geometrie der Raum-Zeit der Maßbestimmung von Kottler, Weyl und Trefftz. *Astron. Nachr.*, 301:135–149, 1980.

[11] S.W. Hawking and G.F.R. Ellis. *The Large Scale Structure of Space-Time.* Cambridge University Press, 1973.

[12] A. Ibrahim and Y. Nutku. Generalized Einstein Static Universe. *Gen. Rel Grav.*, 7:949–958, 1976.

[13] M. Kriele. Space-time: Foundations of general relativity and differential geometry. *Lect. Notes Phys.*, M59:1–432, 1999.

[14] Marc Mars, M. Merce Martin-Prats, and Jose M. M. Senovilla. The $2m \leq r$ property of spherically symmetric static spacetimes. *Phys. Lett.*, A218:147–150, 1996.

[15] Pawel O. Mazur and Emil Mottola. Gravitational condensate stars. 2001.

[16] H. Nariai. On a new cosmological solution of Einstein's field equations of gravitation. *Gen. Rel. Grav.*, 31:963–971, 1999. Originally published in *The Science Reports of the Tohoku University* Series I, vol. **XXXV**, No. 1 (1951), p. 46-57.

[17] H. Nariai. On some static solutions of Einstein's gravitational field equations in a spherically symmetric case. *Gen. Rel. Grav.*, 31:951–961, 1999. Originally published in *The Science Reports of the Tohoku University* Series I, vol. **XXXIV**, No. 3 (1950), p. 160-167.

[18] J. R. Oppenheimer and G. M. Volkoff. On massive neutron cores. *Phys. Rev.*, 55:374–381, 1939.

[19] A. D. Rendall and B. G. Schmidt. Existence and properties of spherically symmetric static fluid bodies with a given equation of state. *Class. Quant. Grav.*, 8:985–1000, 1991.

[20] Pavol Severa. On geometry behind birkhoff theorem. 2002.

[21] H. Stephani. Über Lösungen der Einsteinschen Feldgleichungen, die sich in einen fünfdimensionalen flachen Raum einbetten lassen. *Commun. Math. Phys.*, 4:137–142, 1967.

[22] H. Stephani. *Allgemeine Relativitätstheorie*. VEB Deutscher Verlag der Wissenschaften, 1977.

[23] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt. *Exact solutions of Einstein's field equations*. Cambridge University Press, 2003.

[24] Z. Stuchlík. Spherically symmetric static configurations of uniform density in spacetimes with a non-zero cosmological constant. *acta physica slovaca*, 50:219–228, 2000.

[25] Richard C. Tolman. Static solutions of Einstein's field equations for spheres of fluid. *Phys. Rev.*, 55:364–373, 1939.

[26] S. Weinberg. *Gravitation and Cosmology: Principles and Application of the General Theory of Relativity*. Wiley, 1972.

[27] H. Weyl. Über die statischen kugelsymmetrischen Lösungen von Einsteins ≪kosmologischen≫ Graviatationsgleichungen. *Physikalische Zeitschrift*, 20:31–34, 1919.

[28] D. Winter. Analysis of the cosmological Oppenheimer-Volkoff equations. *J. Math. Phys.*, 41:5582–5597, 2000.

[29] T. Zannias. Spacetimes admitting a three-parameter group of isometries and quasilocal gravitational mass. *Phys. Rev.*, D41:3252–3254, 1990.

Deutsche Zusammenfassung

Die vorliegende Arbeit untersucht statische, kugelsymmetrische Lösungen der Einsteinschen Feldgleichungen mit kosmologischer Konstante. Für die Materie wird eine ideale Flüssigkeit mit vorgegebener monotoner Zustandsgleichung angenommen. Es werden neue Lösungstypen beschrieben.

Bemerkenswert ist, daß die kosmologische Konstante innerhalb dieser Fragestellung lange vernachlässigt wurde.

Das erste Kapitel ist eine Einführung. Sie ordnet die Thematik innerhalb der Gravitationstheorie ein und geht auf Teile vorhandener Literatur ein.

Obwohl schon Weyl [27] diese Lösungen untersuchte und wesentliche Ergebnisse darstellte, wurden sie kaum weiter beachtet. Erst wieder Kriele [13], Stuchlík [24] und Winter [28] untersuchten solche Lösungen.

Im zweiten Teil werden die relevanten Gleichungen hergeleitet und deren mathematische Struktur beschrieben.

Wird für die Materie eine ideale Flüssigkeit angenommen, so erhält man eine Verallgemeinerung der Tolman-Oppenheimer-Volkoff Gleichung mit kosmologischer Konstante (2.16), kurz TOV-Λ Gleichung. Sei nun eine Zustandsgleichung $\rho = \rho(P)$ vorgegeben. Dann bilden die beiden Gleichungen (2.16) und (2.18) ein System von gewöhnlichen Differentialgleichungen in den Funktionen $P(r)$ und $w(r)$.

Für das Vakuum lassen sich die Feldgleichungen einfach lösen. Man erhält die bekannte Schwarzschild-de Sitter Lösung (2.22) für positive kosmologische Konstante bzw. die Schwarzschild-anti-de Sitter Lösung für das negative Vorzeichen. Dies ist die einzige statische, kugelsymmetrische Lösung der Vakuumfeldgleichungen mit nichtkonstanten Gruppenbahnen.

Penrose-Carter Diagramm 28 zeigt die Schwarzschild-de Sitter Raum-Zeit mit dem Ereignishorizont des Schwarzen Loches und dem kosmologischen Ereignishorizont.

Das Penrose-Carter Diagramm 29 zeigt die Schwarzschild-anti-de Sitter Raum-Zeit. In ihr gibt es nur den Ereignishorizont des Schwarzen Loches.

Mit kosmologischer Konstante gibt es eine weitere statische, kugelsymmetrische Lösung der Feldgleichungen für das Vakuum, die Nariai Lösung [16, 17]. Das Volumen der Gruppenbahnen ist hierbei konstant. Einer der neuen Lösungstypen erfordert diese Lösung für den Vakuumteil.

Nach dieser Beschreibung der Gleichungen wird kurz auf deren Newtonschen Grenzwert eingegangen. Wie zu erwarten, ergibt sich die Newtonsche Theorie mit einem zusätzlichen Λ-Term (2.23). Das Newtonsche Analogon der TOV-Λ Gleichung ist die Euler Gleichung der Hydrodynamik im hydrodynamischen Gleichgewicht, wieder mit korrigierten Potential, (2.25).

Der dritte Abschnitt befaßt sich mit den speziellen konstante Dichte Lösungen. Die TOV-Λ Gleichung wird für den einfachen Fall einer Zustandsgleichung $\rho = \rho(P) \to \rho = \rho_0 = \text{const.}$, nämlich homogener Dichte, integriert. Der zusätzlich vorhandene Λ-Term eröffnet neue Lösungen. Die $t = \text{const.}$ Hyperflächen sind bei verschwindender kosmologischer Konstante 3-Sphären. Λ kann nun auch so gewählt werden, daß diese Hyperflächen euklidisch oder hyperbolisch werden. Die Lösungen mit konstanter Dichte sind konform flach, unabhängig von der kosmologischen Konstanten. Mit den im vierten Kapitel eingeführten neuen Buchdahlvariablen kann leicht gezeigt werden, daß dies direkt aus den Feldgleichungen folgt.

Konstante Dichte Lösungen wurden nur bis zu einer oberen Grenze $\Lambda < 4\pi\rho_0$ untersucht [24]. Bis zu dieser Grenze beschreiben die Lösungen ausschließlich Modelle von Sternen. Es existiert immer ein Radius, an dem der Druck verschwindet und an dem eine Vakuumlösung angeschlossen wird. Für diese Modelle läßt sich eine analoge Buchdahl Ungleichung herleiten. Sie besagt, daß der Radius stets zwischen dem Ereignishorizonts des entsprechenden Schwarzen Loches und dem kosmologischen Ereignishorizont liegt. Der kosmologische Ereignishorizont existiert nur im Falle positiver kosmologischer Konstante. Daher haben Modelle mit $\Lambda \leq 0$ im allgemeinen keine obere Schranke für den Radius.

Diese Lösungen können frei von Singularitäten konstruiert werden. Falls $\Lambda \leq 0$ ist dies offensichtlich, da die Außenraumlösungen frei von Singularitäten ist. Für $\Lambda > 0$ muß ein zweites Objekt in die Raum-Zeit gesetzt werden, damit diese Lösungen frei von Singularitäten werden, siehe Penrose-Carter Diagramm 31.

Wird eine größere kosmologische Konstante vorgegeben, so wird die Koordinatensingularität erreicht, noch bevor der Druck verschwindet. An ihr divergiert der Druckgradient, der Druck jedoch bleibt endlich. Da es sich lediglich um eine Koordinatensingularität handelt, können entsprechende Lösungen fortgesetzt werden. In diesem dritten Kapitel werden die verschiedenen Lösungen durch die kosmologische Konstante parametrisiert.

Entspricht Λ der Grenze, so verschwindet der Druck bei der Koordinatensingularität. Nur in diesem Fall erfordert die Innenraumlösung, daß die Nariai Metrik angeschlossen wird. In den anderen Fällen ist es immer die Schwarzschild-de Sitter Lösung, die den Vakuumteil beschreibt. Wird der kosmologische Term nun größer gewählt, dann verschwindet der Druck nach der Koordinatensingularität. Dort ist das Volumen der Gruppenbahnen fallend und man hat den Teil der Schwarzschild-de Sitter Metrik anzuschließen, der die $r = 0$ Singularität enthält, siehe Penrose-Carter Diagramm 32. Eine weitere Vergrößerung führt dazu, daß der Druck nicht mehr verschwinden

kann, er ist strikt positiv. Es ergeben sich Lösungen mit zwei Zentren, die in Fall konstanter Dichte beide regulär sind. Dies sind Verallgemeinerungen des statischen Einstein Universums. Sie haben homogene Dichte aber inhomogenen Druck. Der im ersten Zentrum vorgegebene Zentraldruck fällt monoton bis zum zweiten Zentrum. Bei weiterer Vergrößerung ergibt sich der Einstein Kosmos selbst, um dann deren Verallgemeinerungen mit ansteigendem Druck bis zum zweiten Zentrum zu beschreiben. In Abschnitt 3.4.6 wurde gezeigt, daß sich die Verallgemeinerungen des statischen Einstein Universums mit ansteigendem Druck im ersten Zentrum und diese mit fallendem Druck im ersten Zentrum ineinander überführen lassen.

Als letztes, bei noch größerer kosmologischer Konstante ist der Druck monoton ansteigend und divergiert. Dort hat die Raum-Zeit eine geometrischen Singularität. Diese Lösungen sind unphysikalisch und werden nicht weiter diskutiert. Am Ende dieses Kapitels werden alle konstante Dichte Lösungen in einer Übersicht dargestellt.

Das vierte Kapitel beschäftigt sich zunächst mit der Existenz von globalen Lösungen. Mit neuen Buchdahlvariablen kann diese bis zu einer oberen Grenze der kosmologischen Konstante (4.33) bewiesen werden. Die entsprechenden Lösungen werden beschrieben und mit denen konstanter Dichte verglichen. Die obere Grenze, bis zu der die Existenz einer globalen Lösung gezeigt werden kann, schließt aber die neuen Lösungstypen aus.

Für endliche Lösungen wird die verallgemeinerte Buchdahlungleichung (4.41) hergeleitet. Sie gilt für alle statischen, kugelsymmetrischen Flüssigkeitskugeln deren Dichte nach außen nicht ansteigt. Damit wird gezeigt, daß Lösungen ohne Singularitäten nicht nur im Fall konstanter Dichte konstruiert werden können, sondern daß diese Lösungen für beliebige Zustandsgleichungen diese Eigenschaft besitzen.

Den Abschluß des Kapitels bilden Bemerkungen über die Existenz eines endlichen Radius für eine vorgegebene Zustandsgleichung. Es werden eine notwendige und eine hinreichende Bedingung hergeleitet.

Im Anhang dieser Arbeit werden Ricci und Einstein Tensor sowie einige geometrische Invarianten berechnet. Darüber hinaus sind dort die Penrose-Carter Diagramme dargestellt.

Erklärung

Hiermit erkläre ich, daß ich die vorliegende Arbeit selbstständig und ohne fremde Hilfe verfaßt habe und mich anderer als der angegebenen Quellen und Hilfsmittel nicht bedient habe. Alle Stellen, die wörtlich oder sinngemäß aus Veröffentlichungen entnommen wurden, sind als solche gekennzeichnet.

Golm, 16. Juli 2002

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